EVALUATION OF INFORMATION BUNDLES IN ENGINEERING DECISIONS

A Dissertation

by

NİYAZİ ONUR BAKIR

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

August 2004

Major Subject: Industrial Engineering

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ABSTRACT

Evaluation of Information Bundles

in Engineering Decisions. (August 2004)

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This dissertation addresses the question of choosing the best information alternative in engineering decisions. The decision maker maximizes his expected utility under uncertainty where both the action he takes and the state of the environment determines the payoff earned. The decision maker has an opportunity to gather information about the decision environment *a priori* at a certain cost. There might be different information alternatives, and the decision maker has to determine which alternative offers "better" prospects for improving the decision.

Any decision environment that is characterized by a finite number of outcomes and a discrete probability distribution over the set of outcomes is a lottery. We analyze the value of information on a single outcome and determine the attributes in each piece of information that maximizes its value. Information is valuable when the decision is changed after gathering information. We show that if the number of optimal actions taken under different outcomes scenarios is finite, the decision maker does not require the perfect information. Further, we analyze the relation between the value of information and its determinants, and show a monotonic relation exists for a restricted class of information bundles and utility functions. We use different approaches to evaluate information and analyze the cases where preference reversals occur between different approaches. We observe that *a priori* pricing of information does not necessarily induce the same ranking with the expected utility approach, however both approaches agree

on whether a given piece of information is valuable or not.

The second part of this dissertation evaluates information in both static and dynamic coinsurance problems. In static insurance decisions, we analyze the case where the decision maker gathers information about the severity of the risk events and perform ranking of information bundles in a specific class. In dynamic insurance problems, we make a case study to analyze different physical risks that the production facilities are exposed to. The information in dynamic insurance problems involves more detail with regard to the timing of the multiple risk events. We observe that information on events that pose relatively good scenarios for the decision maker have value, however, their value may diminish as their probability of occurance decreases. The decision maker purchases more information as the profitability of the product increases and less information as the initial wealth increases. Furthermore, the decrease cost of insurance does not necessarily make information more valuable as the value is directly related to the change in the decisions rather than the cost of taking a specific action.

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CHAPTER I

INTRODUCTION

Decision making is the process of identifying, evaluating and choosing alternatives based on the preferences of the decision maker and the limitations that arise in the decision environment. Most decisions are made under uncertainty, and information plays a key role in characterizing the uncertainty in the environment. Gathering information enables the decision maker to reduce the uncertainty and make better decisions. In reality, most useful information does not come for free. The decision maker faces a second decision to make with regard to which information alternative to choose given his current wealth. In this respect, it is vital to build tools to determine the value of information *a priori* and accordingly rank different information based on the underlying preferences of the decision maker.

Information is a collection of events. The decision maker learns that several events occur or not before making the decision. In this respect, any incoming information reduces the size of the outcome space, thus reducing uncertainty. In this study, the decision maker is assumed to know the actual probability distribution governing the outcome space. Furthermore, he is assumed to be an expected utility maximizer, using the actual probability distribution in weighting the possible outcomes. The decision maker may consider many aspects of consumption in the utility function; however in this research, we assume that the decision maker cares only about the monetary value of all items that constitute his total wealth. In other words, the decision maker cares only about money.

This dissertation follows the style of the *IEEE Transactions on Reliability.*

In many engineering models, the decision maker is an expected utility maximizer. However, the utility function is assumed to be linear in most models. Therefore, the attitude of the decision maker towards risk is largely ignored for engineering decision models. This is a big drawback in determining the optimal action in many decision environments because the degree of risk aversion is critical in the ranking of risky alternatives. For example, Ohlson [30] shows that information is a risky alternative and there is no uniform ranking of different information alternatives for each decision maker. A uniform ranking can be established only if we restrict ourselves to a class of utility functions. In this respect, most engineering models are solved for a single type of decision maker, whose attitude towards risk is in reality not very typical of human nature.

A simple model of decision making can be constructed using simple lotteries and giving the decision maker the flexibility to choose among two actions: play or not play. Such a model may not be directly applicable in many decision settings with an infinitely many number of possible outcomes, however the results from this model can be generalized as long as the decision maker chooses among two actions. The computations are relatively straightforward and information can be easily incorporated in these models. Many investment decisions that a firm faces fit the description of either a simple lottery or a generalized lottery, but clearly with only two possible actions. A different lottery is faced after choosing whether to invest or not. The firm may choose to gather some further information to learn the prospects provided by the investment.

Insurance decisions are similar in that the firm may choose to gather information about the possible damage. However, depending on the type of insurance contract, there may be infinitely many possible actions to take. In a coinsurance model, this is clearly the case as the firm has the flexibility to choose what fraction of the risk to transfer to the insurance company. In cap-limit insurance models with a deductible, the size of the action space depends on the number of alternative contracts provided by the insurer. In any case, it is important to identify the partition the outcome space based on optimal actions given a certain outcome. This determines the events that will give the decision maker the most valuable information to improve the insurance decision.

Today, as enterprises are expanding to serve global markets, they are exposed to a variety of operational and hazard risks and are compelled to refine the execution of their core business. It is neither possible nor profitable to eliminate all the risks, as the elimination of risks lead to elimination of many rewarding prospects. Rather, the enterprise should retain risks that are congruent to their core competency and manage them in an effective manner. Insurance decisions are an important component of risk management. Physical risks that the facilities are exposed to are among the most hazardous risks, and their transfer to another party via insurance is vital for the enterprise. However, some physical risks are low probability events with severe impact (e.g., the power shutdown that hit North America in 2003), which may be very costly to insure against. Gathering information reduces uncertainty and helps the enterprises to choose how much coverage to purchase and how much to pay. We study ranking of information in insurance decisions and determine what factors affect the value of information in these decisions.

There are several approaches to evaluate the value of information. Hazen and Sounderpian [13] shows that there is no agreement between different approaches in ranking of information alternatives. The expected utility approach measures the value of information as the difference of expected utility with information and expected utility without information. This difference is always non-negative because the decision maker can never be worse off obtaining information in single person decision environments. This is not the case in decision environments involving multiple decision makers. As in

Arya et al. [5], and Doherty and Thistle [9], information can have negative value when there are conflicting objectives of the decision makers and when one party knows that the other party possesses a piece of information that he may take advantage of. On the other hand, the selling price approach translates the improvement in expected utility by acquisition of information to monetary terms. In this respect, there is no preference reversal between the expected utility and sale price approach. A third approach, which is particularly useful in pricing information is the buying price approach. It assumes that the decision maker pays before acquiring information. Preference reversals are observed between this approach and the other two. The first and essentially the second approach is preferred for computational ease, hence it is easier to identify the relation between the value of information and it determinants. It is not possible to obtain a general closed form expression for the buying price of information. In this document, we address how and when preference reversals occur and identify the relation between the value of information and its key determinants.

This dissertation is organized as follows. In Chapter II, we review different approaches to determine the value of information and provide discussion on the previous literature on the topic. In Chapter III, we analyze the value of information in lotteries and obtain results on the relation between the value of information and its determinants. In particular we analyze how the degree of risk aversion and the level of initial wealth affects the value of information in lotteries. In Chapter IV, we first establish the relation between the expected utility approach and buying price approach. Some examples to illustrate the preference reversal are provided. Then, we analyze how the degree of risk aversion affects maximum price that the decision maker accepts for information. In Chapter V, we visit a choice information problem for insurance decisions. The problem is static in nature, in that the hazard that the decision maker is exposed to occur at one point in time. We consider ranking of two different classes of information alternatives. In Chapter VI, we consider a dynamic enterprise risk management problem. Two types of risk events hit a facility over time, causing a reduction in the overall capacity. Reduction of overall capacity exposes the facility to risk of losing sales. The enterprise considers insurance for such business interruption risk. We analyze different information alternatives and determine their value to the enterprise in insurance decisions. Finally, in Chapter VII, we state our concluding remarks and outline possible research directions.

CHAPTER II

REVIEW OF THE LITERATURE AND PREVIOUS RESEARCH

The concept of information evaluation has been a part of economic and statistical decision theory for some time. It has long been accepted that decisions made under uncertainty can be improved (in the sense of higher expected utility) by acquiring information. Previous literature concentrated mainly on determination of optimal amount of information that the decision maker should learn before taking the necessary action. Additional information might induce the decision maker to update the initial decision to achieve a higher expected utility. However, not all the useful information may be available. The selection becomes interesting when one considers the problem with cost and feasibility concerns.

Three different approaches for including information in decision making have been considered in the literature. The most popular approach is to obtain data by sampling a population. Sampling information is valuable when the decision involves some characteristic of a population. The second approach is to gather information in the form of signals. In this scenario, decision maker can obtain data that serve as a signal about the actual state of the environment. The final approach is to consider information as a collection of events. The decision maker determines certain events have actually occured before taking the action. The mathematical structure that is used to represent information is a σ-algebra in this case.

The decision maker's expected utility with no information and with perfect information are two benchmarks that can be used to evaluate performance with partial information. The first benchmark measures how well the decision maker does on the average when he takes a single action in all the possible states, whereas the second one measures how much better he could do by eliminating all uncertainty at the decision epoch. The difference between these two benchmarks represents the expected gain in utility from knowing exactly which state obtains before the action is taken. Perfect information gives the opportunity to select the best action in every possible state.

II.1. Quantifying Information for Decision Making under Uncertainty

Consider a probability space $(\Omega, \mathcal{M}, \mu)$. Let $\mathcal{A}$ denote a discrete action space and $u : \mathbb{R} \to \mathbb{R}$ be a utility function. We attribute a single outcome to each action and state combination. This relation is mathematically represented by a payoff function $p : \Omega \times \mathcal{A} \to \mathbb{R}$. The payoff function maps each state and action combination to a monetary value as an outcome. The utility function represents the preferences of the decision maker by assigning a numerical value to each possible outcome. The ranking of different outcomes by the utility function is inherited from the underlying preference relation. However, the ranking of risky alternatives requires additional axioms.

Definition 1 *A lottery $L : \Omega \to \mathbb{R}$ is a $(\Omega, \mathcal{M}, \mu)$ measurable random variable with a distribution function $F_L(x) = \mu \circ L^{-1}(x)$. (i.e., a random variable that maps each risky alternative to an outcome.)*

Let $\mathcal{L}$ be the space of lotteries. Following Mas-Colell et al. [24], we assume that the decision maker has a preference relation, $\preceq$, over $\mathcal{L}$ that satisfies the following axioms:

Axiom of Rationality The preference relation, $\preceq$, is rational if it is both complete and transitive.

Axiom of Continuity The preference relation, $\preceq$, is continuous if for any $L, L', L'' \in \mathcal{L}$, the sets $\{\alpha \in [0, 1] : \alpha L + (1 - \alpha)L' \succeq L''\} \subset [0, 1]$ and $\{\alpha \in [0, 1] : \alpha L + (1 - \alpha)L' \preceq L''\} \subset [0, 1]$ are closed.

Axiom of Independence The preference relation, $\preceq$, satisfies the independence axiom if for all $L, L', L'' \in \mathcal{L}$, and $\alpha \in (0,1)$ we have:

$$L \succeq L' \qquad \text{if and only if} \qquad \alpha L + (1-\alpha)L'' \succeq \alpha L' + (1-\alpha)L''$$

The independence axiom has been challenged in the literature both theoretically and experimentally, but we will not pursue such a direction in this research and we assume this axiom holds. It is a strong axiom that lies in the center of expected utility theory because it permits the expected utility representation of the preferences over the lotteries.

Definition 2 *The utility function $U : \mathcal{L} \to \mathbb{R}$ has an expected utility form if it satisfies $U(L) = \int u(x) dF_L(x)$.*

The utility function $U : \mathcal{L} \to \mathbb{R}$ has the expected utility form if and only if it is linear. The expected utility form is preserved by increasing linear transformations. The decision maker is an expected utility maximizer as his preferences over the lotteries are represented with a utility in the expected utility form. The following theorem sets forth this important conclusion.

Theorem 1 (Expected Utility Theorem): *Suppose that the rational preference relation, $\succeq$, on the space of lotteries $\mathcal{L}$ satisfies the continuity and independence axioms. Then $\succeq$ admits a utility representation of the expected utility form. That is, the following holds:*

$$L \succeq L' \qquad \textit{if and only if} \qquad \int u(x) dF_L(x) \geq \int u(x) dF_{L'}(x)$$

Each decision maker would like to choose the action that maximizes utility in each state. However, without knowledge of the states of world, he can choose only a single action that will *maximize his expected utility over all states.* Let a^* denote the action taken under this criterion. On the other hand, if the decision maker obtains perfect information a priori, he will have the flexibility of choosing the best action in each state. Hence, the expected value of perfect information is

$$\text{EVPI} = \int \max_{a_i \in \mathcal{A}} u(p(\omega, a_i)) \; d\mu(\omega) - \int u(p(\omega, a^*)) \; d\mu(\omega).$$

Statistical decision theory mainly focuses on information that can be gathered by sampling. Sampling cost is assumed to be an increasing function of the sample size, so the optimal sample size has to be chosen. The action is taken after the outcome from the sample is observed. In this sense, the action is a function of the outcome from the sample. The decision maker chooses an optimal response to each outcome. Following Tummala [37] and Raiffa and Schlaifer [32], the problem structure is as follows. Let k be the sample size, and let S_k be the set of all the outcomes of the sample. We define $X_k : \Omega \to S_k$ be the random variable representing the outcome of the sample of size k. The decision function, $d_k : S_k \to \mathcal{A}$, extablishes the functional relationship between the outcome of the sample and the action taken. Let $\mathcal{D}_k$ denote the space of decision functions when the sample size is k and $V_k : \Omega \times \mathcal{D}_k \to \mathbb{R}$ be the expected value function. When the decision function $\bar{d} \in \mathcal{D}_k$ is chosen, the expected value of sampling in state ω is given by

$$V_k(\omega, \bar{d}) = \int u(p(\omega, \bar{d}(s))) \; dP(s|\omega)$$

where $P(s|\omega)$ is the probability of a sample outcome of s given the actual state ω. Since the decision maker does not know the actual state, he computes the expected value of sampling when decision function $\bar{d}$ is chosen, as

$$\mathbb{E}V_k(\bar{d}) = \int V_k(\omega, \bar{d}) \; d\mu(\omega)$$

Define $d_k^* = \text{argmax}_{d \in D_k} \mathbb{E}V_k(d)$, where $\mathbb{E}$ denotes the expectation operator. Then the expected value of sampling with sample size of k is,

$$\mathrm{EVSI}(k) = \mathbb{E}V_k(d_k^*) - \int u(p(\omega, a^*))\ d\mu(\omega) - c_k$$

where c_k denotes the fixed cost of sampling of size k. The optimal sample size is found by maximizing EVSI(k) over k.

So-called *information systems* are another form of modeling information acquisition. They are a collection of potential signals that can be observed in the environment. The decision maker observes signals before making the decision. These signals are outcomes of some random variables. Suppose that the decision maker considers purchasing information system $\mathcal{I} = \{I_1, ..., I_n\}$ where I_i is a random variable that is available for observation under information system I. Information systems have been studied by Marschak and Radner [23], Marschak and Miyasawa [22], Gjesdal [11], Miller [28] and many others. The generally accepted definition of the value of information system $\mathcal{I}$ is

$$\mathbb{E}V(I) = \mathbb{E}\left[\max_{a \in \mathcal{A}_{\mathcal{I}}} \mathbb{E}\left[u(p(\omega, a))|I\right]\right] - \mathbb{E}\left[u(p(\omega, a^*))\right] - c_I$$

where c_I is the fixed cost of information system I and $\mathcal{A}_{\mathcal{I}}$ is the set of feasible actions under information system $\mathcal{I}$ that the decision maker has the flexibility to take at the time of making the decision. It can be observed that the approach to measure the value of information is similar to the previous one in that expected a priori gain from purchasing information should exceed expected gain without information plus the cost of information. The value of information is determined by the utility of the decision maker, flexibility in taking action, feasibility of an action with respect to the information system, the probability law that governs the states of the world, initial level of wealth and the cost of obtaining information.

The third approach to measure the value of information has been formalized in the context of general equilibrium in economic theory. Radner [31] incorporated uncertainty in the general equilibrium model developed by Arrow [4] and Debreu [8] in that he considered different economic agents with different information. He considered information as a collection events represented mathematically by a σ-algebra. Allen [1] studied the mathematical properties of these information structures and showed that the value of information mapping is continuous with respect to the Hausdorff metric on σ-algebras. Furthermore, Allen [2] treated information as a commodity with a given price and formulated the problem of information acquisition. A dynamic model for information acquisition is offered in Allen [3], where the firms treat R&D operations as information acquision activities with uncertainty incorporated regarding the consequence of R&D investment. Following her notation, we describe the problem as follows: Consider a complete probability space $(\Omega, \mathcal{M}, \mu)$. An information bundle is a sub-σ-field of $\mathcal{M}$. The set of all complete sub-σ-fields of $\mathcal{M}$ is denoted as $\mathcal{M}^*$. In Allen's model, $\mathcal{U} = \{v \in C^0(\mathbb{R}^l_+, \mathbb{R})$: v is strictly monotone and strictly concave$\}$; i.e., $\mathcal{U}$ is the basic set of continuous utility functions.

Definition 3 *The initial endowment, $e \in \mathbb{R}^l_{++}$, is the amount of commodities that a decision maker owns before trade.*

Let $\Delta = \{q \in \mathbb{R}^l_{++} | \Sigma^l_{j=1} |q_j| = 1\}$ be the $(l-1)$-dimensional open unit simplex in $\mathbb{R}^l$. The initial endowment is a non-zero vector. Allen defined the value of information as [1],

Definition 4 *If $U : \Omega \to \mathcal{U}$ (where there is a compact $K \subset \mathcal{U}$ such that $U(\omega) \in K$ μ-a.s.), $e \in \mathbb{R}^l_{++}$ and $q \in \Delta$, define the value (in utility terms) of the information provided by sub-σ-field $\mathcal{G}$ of $\mathcal{M}$ to be,*

$$
\begin{aligned}
V(\mathcal{G}; U, e, p) = \ & max\{\textstyle\int_\Omega u(p(\omega, a(\omega)))d\mu(\omega) | a : \Omega \to \mathbb{R}^l_+ \\
& \qquad\qquad \text{is } \mathcal{G}\text{-measurable and } q \cdot a(\omega) \leq q \cdot e \ \mu\text{-a.s.}\} \\
& - \ max\{\textstyle\int_\Omega u(p(\omega, a(\omega)))d\mu(\omega) | a : \Omega \to \mathbb{R}^l_+
\end{aligned}
$$

$$\textit{is constant and satisfies } q \cdot a(\omega) \leq q \cdot e\}$$

where the utility function is state dependent.

Essentially, the second and third approaches derive from statistical decision theory. One can observe that all three approaches are Bayesian in nature as the decision maker updates the probability distribution on the states of the world after obtaining information. Another quick observation is to make an immediate connection between the second and third approaches. The conditional expectation with respect to a collection of signals is the same as conditional expectation with respect to σ-algebra generated by the random variables which generate those signals. In this document, we consider information as collection of events, hence the third approach will be employed to measure the value of information in what follows.

II.1.1. Comparison of Information Bundles

It is important to be able to compare different information bundles. Several different approaches have been taken in the literature. One approach is to compare information bundles by a notion of distance in a metric space. In this approach, $\mathcal{M}^*$ is treated as a metric space and several equivalent metrics are used to define a topology. This sort of mathematical structure is particularly useful to address the questions of convergence of information bundles. Let $\mathcal{F}$ and $\mathcal{G}$ be two elements in $\mathcal{M}^*$. Boylan [6] proposed the following metric:

$$d_1(\mathcal{F},\mathcal{G}) = \sup_{\mathcal{F}\in\mathcal{F}} \inf_{\mathcal{G}\in\mathcal{G}} \mu(\mathcal{F}\Delta G) + \sup_{G\in\mathcal{G}} \inf_{F\in\mathcal{F}} \mu(F\Delta G)$$

This metric involves two pieces assuming symmetric interpretations. Note that, $\mu(F\Delta G)$ piece is itself a metric on the space of $\mathcal{M}$ where each element of $\mathcal{M}$ is an event. Extending the notion of using probability on the symmetric difference between two sets as a measure of distance between two events, two pieces of the metric are obtained. The first term in the expression for $d_1(\mathcal{F},\mathcal{G})$ measures how closely $\mathcal{G}$ resembles $\mathcal{F}$ and the second term measures how closely $\mathcal{F}$ resembles $\mathcal{G}$. Rogge [33] formulated an equivalent metric by using the conditional expectations operator $E[\cdot|\mathcal{F}]$ as follows

$$d_2(\mathcal{F},\mathcal{G}) = \sup_{f\in\Theta}\|\mathbb{E}[f|\mathcal{F}]-\mathbb{E}[f|\mathcal{G}]\|_1$$

where Θ is the set of all $\mathcal{M}$-measurable functions.

A third metric is defined by Van Zandt [38] in a separable metric space (X,d), in which the $\mathcal{M}$-measurable functions take value. Let $\theta(f,g) = \inf\ \{\varepsilon > 0|\mu\{\omega \in \Omega \mid d(f(\omega),g(\omega)) > \varepsilon\} < \varepsilon\}$ be a metric which induces a notion of convergence in measure and generates a topology on the equivalence classes of $\mathcal{M}$-measurable functions. This metric is a distance between two $\mathcal{M}$-measurable functions in that they are close if they don't differ much except on a subset of Ω with an insignificant measure. The Hausdorff distance between two information bundles is also characterized by comparing their respective measurable actions as follows:

$$d_3(\mathcal{F},\mathcal{G}) = \max\{\sup_{f\in\mathcal{M}_\mathcal{F}}\inf_{g\in\mathcal{M}_\mathcal{G}}\theta(f,g), \sup_{g\in\mathcal{M}_\mathcal{G}}\inf_{f\in\mathcal{M}_\mathcal{F}}\theta(f,g)\}$$

where $\mathcal{M}_\mathcal{F}$ and $\mathcal{M}_\mathcal{G}$ are the sets of $\mathcal{F}$ and $\mathcal{G}$ measurable functions respectively. This metric establishes a nice connection between two information bundles. The additional value that each information bundle provides to the decision-maker is an enlarged set of feasible policies. The equivalence of the last metric says that if two bundles give information about very similar events, then the feasible policies are also similar.

These metrics have been studied extensively by Boylan [6], Rogge [33], Landers and Rogge [20] and Van Zandt [38]. Rogge showed that $d_1(\mathcal{F},\mathcal{G}) \leq d_2(\mathcal{F},\mathcal{G})$. Landers and Rogge established the equivalence between these two metrics and showed that $d_2(\mathcal{F},\mathcal{G}) \leq 8d_1(\mathcal{F},\mathcal{G})$. Van Zandt extended the equivalence result and showed that the first and the third metrics are equivalent.

Theorem 2 Van Zandt ([38], 1993): $min\{diam(X,d)/2, d_1(\mathcal{F},\mathcal{G})\} \leq d_3(\mathcal{F},\mathcal{G}) \leq 4d_1(\mathcal{F},\mathcal{G})$.

This equivalence relation makes the connection between similarity of information bundles and the similarity of informationally feasible decision rules. Each element of $\mathcal{M}_{\mathcal{F}}$ can be interpreted as an informationally feasible policy given $\mathcal{F}$. The additional value that each information bundle provides to the decision maker is an enlarged set of feasible policies. The equivalence of $d_1(\mathcal{F}, \mathcal{G})$ and $d_3(\mathcal{F}, \mathcal{G})$ says that if two bundles give information about very similar events, then the feasible policies are also very similar. The decision maker is restricted with the amount of information he has when he chooses among the set of feasible policies. It is not possible to implement a complex policy with very restricted information as the information bundle does not form a nice partition of the state space. In the case of no information, $\mathcal{F} = \{\Omega, \emptyset\}$, the set of feasible policies are only the constant functions. On the other hand, when the decision maker can observe full information about the states of the world, he can follow the best policy in any state. This full information case is captured in EVPI.

Similar strategies can be implemented under similar information bundles. This will be very useful in modeling the choice of information in that the decision maker may be able to substitute an information bundle with something similar enough when financial limitations arise. In a dynamic situation where information packages can be mathematically viewed as a filtration, $\{\mathcal{F}_t\}_{t\geq 0}$, increasing to a σ-field $\mathcal{F}_\infty$, Boylan [6] showed the following:

Theorem 3 Boylan ([6], 1971): *Let $\mathcal{F}_t$, $t = 1, 2, ..., \infty$ be subfields of $\mathcal{F}$ with $\mathcal{F}_t$ increasing or decreasing to $\mathcal{F}_\infty$ and $lim_{t\to\infty}\, d_1(\mathcal{F}_t, \mathcal{F}_\infty) = 0$. Then the functions $E[f|\mathcal{F}_t]$, $t = 1, 2, ...,$ such that $\|f\|_\infty \leq 1$ are strongly equiconvergent.*

Theorem 4 Boylan ([6], 1971): *Let $\mathcal{F}_t$, $t = 1, 2, ...,$ be a sequence of subfields with the property that $lim_{s,t\to\infty} d_1(\mathcal{F}_s, \mathcal{F}_t) = 0$. Then there exists a subfield $\mathcal{H}$ such that $lim_{t\to\infty} d_1(\mathcal{F}_t, \mathcal{H}) = 0$. Moreover, for every f integrable, $E[f|\mathcal{F}_t]$ converges in measure to $E[f|\mathcal{H}]$.*

Therefore, it can be suggested that after some point in time implementation of some strategy will be very similar as we get more information. Accordingly, the decision maker can stop gathering information after some time for cost saving purposes. However, if there are some extreme events with financially severe consequences, the decision maker may assume the risk of facing an extreme event without any prior detection after stopping to gather information. In this respect, the time to stop gathering information depends on the degree of risk aversion of the decision maker.

Allen [1] shows the following continuity results,

Theorem 5 Allen ([1], 1983): *If U, e and p are as in the definition above and if $\mathcal{G}_n \to \mathcal{G}$ in the metric topology of $\mathcal{F}^* = \{$space of equivalence classes of sub-σ fields of $\mathcal{F}$ with respect to the metric d_1 $\}$, then $V(\mathcal{G}_n; U, e, p) \to V(\mathcal{G}; U, e, p)$.*

Corollary 1 Allen ([1], 1983): *If $\{U_n\}$ is a sequence of measurable functions defined on Ω such that $U_n \to U$ in $\mathcal{U}$ topology (pointwise) on a subset Ω of measure one and $\{U_n\}$ sequence is uniformly bounded, if $e_n \to e \in \mathbb{R}^l_{++}$, if $p_n \to p \in \Delta$ and if $\mathcal{G}_n \to \mathcal{G}$ in the $\mathcal{F}^*$ topology, then $V(\mathcal{G}_n; U, e, p) \to V(\mathcal{G}; U, e, p)$.*

This continuity result was extended by Van Zandt [38].

Corollary 2 Van Zandt ([38], 1993): *The value of information map is uniformly continuous with respect to the Hausdorff metric d_1.*

These results suggest that similar information bundles are also close in value for a decision maker. Thus, if the decision maker acquires an information bundle that consists of similar events with some target information bundle that is not available, then he can expect a similar performance of his decision. This fact has practical importance because in reality information sources are limited and decisions are usually made missing critical information. The continuity result holds for any decision maker, however it does not guarantee identical ranking of two information bundles for different decision makers given that they are sufficiently similar (i.e., decision makers with different utility functions and/or initial wealth levels).

II.1.2. Information Pricing

The difference in expected utilities provides a good benchmark to evaluate the value of information. However, it does not address the question of how much a decision maker is willing to pay for the information when the utility function is not separable in the cost of information. The question of information pricing has been thoroughly studied by LaValle [17] [18] [19]. His model is important as it provides a nice connection between the shape of the utility function and the value of information. The relation between risk taking behavior and the price that the decision maker is willing to pay for information is established. Roughly speaking, information bundles with higher prices rank higher in preference ordering of the decision maker. In his model, $p : \Omega \times \mathcal{A} \to \mathbb{R}$ similarly defines the state and action dependent payoff. The cost of obtaining information is also state dependent, $c_{\mathcal{F}} : \Omega \to \mathbb{R}$. The utility function ranks payoffs, $u : \mathbb{R} \to \mathbb{R}$. Let $a^* = \mathrm{argmax}_{a \in \mathcal{A}}\ \mathbb{E}[u(p(\omega, a) + y)]$. Then, a^* is the optimal action without any information. LaValle defined two different prices for information, one for the seller side, one for the buyer side.

Definition 5 *The certainty equivalent, $c(L, u)$, of the lottery L is the amount of money for which the decision maker is indifferent between the sure outcome $c(L, u)$ and the lottery L; that is, $u(c(L, u)) = \mathbb{E}U$ where $\mathbb{E}U$ is the expected utility from the lottery.*

Definition 6 LaValle ([17], 1968): *The seller price of information bundle $\mathcal{F}$ with initial endowment y and cost function $c_{\mathcal{F}}$ is the unique number $S[\mathcal{F}, y, c_{\mathcal{F}}]$ defined by,*

$$u(y + S[\mathcal{F}, y, c_{\mathcal{F}}]) = \mathbb{E}[max_{a \in \mathcal{A}_{\mathcal{F}}} \mathbb{E}[u(p(\omega, a) + y - c_{\mathcal{F}}(\omega))|\mathcal{F}]]$$

Definition 7 LaValle ([17], 1968): *The buyer price of information bundle $\mathcal{F}$ with initial endowment y and cost function $c_{\mathcal{F}}$ is the unique number $B[\mathcal{F}, y, c_{\mathcal{F}}]$ defined by,*

$$u(y) = \mathbb{E}[max_{a \in \mathcal{A}_{\mathcal{F}}} \mathbb{E}[u(p(\omega, a) + y - c_{\mathcal{F}}(\omega) - B[\mathcal{F}, y, c_{\mathcal{F}}])|\mathcal{F}]]$$

where $A_{\mathcal{F}}$ denotes the actions which are $\mathcal{F}$-measurable. The seller price is the maximum amount of additional wealth that the decision maker is willing to forgo for purchasing information bundle $\mathcal{F}$. It is the difference between the certainty equivalent of the risky information gathering activity and the current level of wealth. On the other hand, the buyer price is the maximum amount of additional cost that the information bundle could have in order to make the investment equally desirable with today's asset position. They don't have be equal. Both can be negative. LaValle shows that the following relation holds:

Theorem 6 LaValle ([17], 1968):

(1) $S[\mathcal{F}, y, c_{\mathcal{F}}] = B[\mathcal{F}, y + S[\mathcal{F}, y, c_{\mathcal{F}}], c_{\mathcal{F}}]$

(2) $B[\mathcal{F}, y, c_{\mathcal{F}}] = S[\mathcal{F}, y - B[\mathcal{F}, y, c_{\mathcal{F}}], c_{\mathcal{F}}]$

The first equality suggests that the amount of money required to forgo the decision to purchase information should be the same as the amount of increase in the cost of information that makes the decision maker indifferent between purchasing the information and the new asset position. The second equality has the converse implication. It suggests that the amount of increase in the cost of information that makes the decision maker indifferent between purchasing the information and the original asset position is the same as the amount that the decision maker requires to forgo the purchase of information in the new reduced asset position. LaValle calculates the prices in lotteries where there is one action in each lottery. However, in simple lotteries, the decision maker always has the flexibility to skip the lottery, and this will affect the functional form of the prices of the lotteries.

II.2. Determinants of the Value of Information and Ranking of Information Bundles

Since the utility functions of different individuals may be dissimilar, ranking of information bundles in terms of preference could be quite different. However, it is possible to come up with a common preference ordering on information bundles if we restrict ourselves to a certain collection of utility functions. Ohlson [30] showed that there exists a common ranking of information bundles in a portfolio selection problem if the probabilities of returns are “small risk”. The implication behind this is that moments of degrees greater than two can be neglected. When returns have “small risk” probabilities, Samuelson [35] showed that the mean-variance approach in financial analysis is justified within the theory of expected utility in that the mean-variance approach produces consistent results with the axioms of expected utility maximization. This allows Markowitz [21] type of portfolio selection analysis possible, and Ohlson’s results on optimal portfolio mix turn out to be independent of the particular utility function and the initial wealth.

As shown in Hilton [14] and Gould [12], the value of information does not in general have a monotonic relationship with any of its determinants: action flexibility, risk aversion, initial wealth and the degree of uncertainty. However, LaValle [17] and Willinger [39] obtained monotonicity and invariance results with respect to one of the determinants for a restricted class of utility functions and a particular class of decision settings. Under certain assumptions, Merkhofer [27] performed sensitivity analysis of the value of information with respect to different decision parameters, given action flexibility for quadratic utility functions. Such relationships are very useful especially for a rich class of engineering problems where the utility function of the decision maker is generally not known with certainty. For example, in reliability problems, several different and relevant objective functions are used for very similar decision settings.

Accordingly, the ranking of information bundles could depend on the objective function used unless some general form of relationship or a preference ranking is verified.

Preference ranking reversals are possible when we use different approaches to quantify information value. Hazen and Sounderpian [13] showed that the selling prices and difference in expected utility are ordinally equivalent measures of information value whereas the buying prices may offer preference ranking reversals. The buying price has practical significance in that it measures how much a decision maker is willing to pay to purchase information. The difference in expected utility approach illustrates how much the decision maker is expected to be better off by incorporating more information. In this document, we show when the preference reversals occur and discuss the potential causes for preference reversals. We analyze a simple lottery for this purpose as it is the basic building block of decision making under uncertainty, and it helps us develop an approach for modeling risky alternatives in an easy and concise manner. Every decision making problem with finite number of risky outcomes can be reduced to a simple lottery. Furthermore, the results on simple lotteries can easily be extended to decision settings where the outcomes are realized as a value of a continuous variable.

Intuitively, increase in the degree of risk aversion leads to an increase in the value of information. The reason behind this is that elimination of uncertainty to some extend should be valued more by the risk averse decision maker. However, this line of reasoning ignores the fact that information gathering is itself a risky activity, as the decision maker does not know what the exact information he will learn in advance. In this study, the relation between the value of information and degree of risk aversion is established in simple lotteries under different circumstances.

II.3. Applications of Information Valuation in the Operations Research Literature

The operations research models generally address the impact of information on decision making either through consideration of perfect information or realization of random variables. Howard [15] [16], considered a decision maker assuming partial information through observation of a random variable that has a direct impact on expected utility. Mehrez and Sethi [25] and Mehrez and Stulman [26] considered information purchasing in project selection problem. Their model offers a fixed information system available for each project, and they did not consider different information alternatives for each possible project. Recently, Çetin, Jarrow, Protter, Yıldırım [7] studied a credit risk model under partial information where they consider information as a collection of events. There seems to be a promising path for research for enrichment of operations research models with incorporation of different information structures.

In many operations research models, choice of information bundles among many alternatives has not been addressed. Generally, there are two approaches to incorporate information. One, generally appearing in supply-chain literature, models a contractual relationship between two parties with conflicting objectives and asymmetric information. The value of information is realized through improved performance of contracts via information sharing. The second, as we discussed above, focuses on the question of acquisition of single information bundle or as in filtering theory collection of information bundles that is dynamically enriched over time, without any significant interest to determine the best among several different information sources.

II.4. Value of Information in Insurance Decisions

Insurance is one of the well known mechanisms to transfer risk between two decision makers. Insurance decisions are usually made under asymmetric information conditions. Most of the earlier work on the impact of information on insurance decisions has focused on design of insurance contracts under asymmetric information. Rothschild and Stiglitz [34], Wilson [40], Miyazaki [29], Spence [36] and others have characterized the equilibrium contracts in insurance markets and analyzed the effect of adverse selection on equilibrium. The value of information is not necessarily positive in decision settings that involve more than one decision maker. Doherty and Thistle [9] showed that the insured benefits from acquiring information as long as it is not also observed by the insurer. Such information can be used to the advantage of the insured as the insurer cannot adjust his action in light of this information.

The earlier work does not address the question of what information should be purchased to improve insurance decisions. In the following chapters, the ranking of a certain class of information bundles is established for a decision maker with a specific utility function and the factors that affect the value of information is analyzed. Further, the relation between the amount of coinsurance and the type of information purchased is illustrated. This problem arises in every industry that faces risk of business interruption due to physical risk events. The firm can benefit from collecting information about the intensity of risk events and the severity of risk events. It is important to determine the the desirable information bundle that will facilitate and improve the insurance decisions against business interruption risks given cost considerations. To this end, a simple risk model is considered and the value of information in business interruption risk problems is analyzed.

CHAPTER III

VALUE OF INFORMATION IN LOTTERIES

We consider a lottery setting where the decision maker has to decide whether to play or not. The lottery has prospects of monetary outcomes either positive or negative that the decision maker has to accept. If the decision maker chooses not to play, then there is neither a monetary gain nor a loss. The decision maker is an expected utility maximizer. In this work, we restrict the available information bundles to certain forms of events. The information about the events is made available by a mediator. Both the mediator and the decision maker are assumed to know the probability distribution on the outcomes. Initially, the mediator conveys the information about the true probability distribution and provides some choices of information bundles that are available to the gambler. We wish to see which information bundles are desirable for the gambler.

III.1. Value of Information

Consider a random variable L with a discrete probability law and a finite number of outcomes. This is a typical lottery situation, where the possible actions are either "play" or "skip" the lottery.

Definition 8 *A simple lottery, $L : \Omega \to \mathbb{R}$, is a real valued random variable with a finite number of outcomes and a discrete probability law.*

The decision maker plays the lottery if the expected utility from playing is greater than utility of skipping the lottery. The number of possible outcomes of the lottery is n. The initial wealth of the decision maker is denoted by W. Let $\Pi = (\pi_1, ..., \pi_n)$

denote the vector of outcomes [1] with $F = (p_1, ..., p_n)$ as the probability distribution over the outcomes (i.e., $p_i = \mu(\pi_i)$). Furthermore, let $p(\pi, a)$ be the payoff to the decision maker when action a is taken and outcome π from the lottery is observed. Note that, $p(\pi, \text{"skip"}) = 0$ and $p(\pi, \text{"play"}) = \pi$ for all $\pi \in \Pi$. Suppose the lottery owner provides information about events of the form E_i , where $E_i = \{\text{outcome } \pi_i \text{ occurs}\}$. This particular form of events induces a simple partitioning of the outcome space into two. Accordingly, the decision maker learns *a priori* whether E_i occurs or not. This gives the flexibility of reconsidering the original decision on the basis of incoming information. Since the value of information is computed *a priori*, the maximum expected utility obtained in each event is weighed by their probability of occurance. Let $\mathbb{E}\mathrm{U}$ denote the expected utility earned playing the lottery. We wish to determine the value of information regarding E_i, $1 \leq i \leq n$. The value of information is simply calculated by,

$$\begin{aligned} \mathbb{E}\mathrm{U}_i = P(E_i) \cdot \max\{0, \mathbb{E}[u(p(\pi, \text{"play"}))|E_i]\} + \\ P(E_i^c) \cdot \max\{0, \mathbb{E}[u(p(\pi, \text{"play"}))|E_i^c]\} - \max\{0, \mathbb{E}\mathrm{U}\} \end{aligned} \tag{3.1}$$

It is easy to see that the value of information, 3.1, is nonnegative. The decision maker is never worse off by obtaining more information. We will first obtain an expression for $\mathbb{E}\mathrm{U}_i$ for a risk neutral decision maker and then generalize the result.

III.1.1. Risk Neutral Decision Maker

Assume the decision maker has a linear utility function of the form $u(x) = ax + b$, $a > 0$. Linear utility functions are used to model the preferences of a risk neutral decision maker. The following proposition yields a simplified expression for $\mathbb{E}\mathrm{U}_i$ when

[1]The underlying probability space is $(\Pi, \mathcal{F}, \mu)$. $\mathcal{F}$ is a σ-algebra generated by discrete events and μ is a disrete probability distribution.

the decision maker is risk neutral.

Proposition 1 *For a risk neutral decision maker,*

$$\mathbb{E}\mathrm{U}_i = \begin{cases} |a\pi_i p_i| & : \quad (\mathbb{E}\mathrm{U} - b) \cdot \pi_i < 0 \\ max\ (0, |a\pi_i p_i| - |\mathbb{E}\mathrm{U} - b|) & : \quad (\mathbb{E}\mathrm{U} - b) \cdot \pi_i \geq 0 \end{cases}$$

Proof: First, $\mathbb{E}[u(p(\pi, \text{"play"}))|E_i] = a\pi_i + b$ and $\mathbb{E}[u(p(\pi, \text{"play"}))|E_i^c] = \frac{\mathbb{E}\mathrm{U} - (a\pi_i + b) \cdot p_i}{1 - p_i}$. Then substituting these, the expression becomes

$$\mathbb{E}\mathrm{U}_i = p_i \cdot \max\ \{b, a\pi_i + b\} + (1 - p_i) \cdot \max\ \{b, \frac{\mathbb{E}\mathrm{U} - (a\pi_i + b) \cdot p_i}{1 - p_i}\} - \max\ \{b, \mathbb{E}\mathrm{U}\}$$
$$= \max\ \{p_i \cdot b, (a\pi_i + b) \cdot p_i\} + \max\ \{(1 - p_i) \cdot b, \mathbb{E}\mathrm{U} - (a\pi_i + b) \cdot p_i\} - \max\ \{b, \mathbb{E}\mathrm{U}\}.$$

We have four cases to consider.

Case-1: If $(\mathbb{E}\mathrm{U} - b) < 0$, $\pi_i > 0$, then $\mathbb{E}\mathrm{U}_i = a\pi_i p_i$.

Case-2: If $(\mathbb{E}\mathrm{U} - b) \leq 0$, $\pi_i < 0$, then $\mathbb{E}\mathrm{U}_i = \max\{0, \mathbb{E}\mathrm{U} - b - a\pi_i p_i\}$.

Case-3: If $(\mathbb{E}\mathrm{U} - b) \geq 0$, $\pi_i > 0$, then $\mathbb{E}\mathrm{U}_i = \max\{0, a\pi_i p_i + b - \mathbb{E}\mathrm{U}\}$.

Case-4: If $(\mathbb{E}\mathrm{U} - b) > 0$, $\pi_i < 0$, then $\mathbb{E}\mathrm{U}_i = -a\pi_i p_i$.

These cases can be combined easily to obtain the result. □

The proposition states that obtaining information about any outcome which induces the decision maker change his decision is valuable. If we assume that the lottery is not worth playing in the no information case, one is expected to be better off with a piece of information about any positive outcome. This is intuitive because the decision maker has the opportunity to change the optimal action after observing a positive outcome. In such a case, the second piece of the argument says that the information about a negative outcome π_i might have positive value either when it is highly likely to occur or when its payoff is significantly high in absolute value. Note that the decision maker updates the original decision in case such an event does not occur. The contribution $\pi_i \cdot p_i$ of such an outcome to the expected utility with no information should be dominant so that the sign of the expected utility changes when such an outcome fails to occur. The level of initial wealth has no impact on the value of information.

III.1.2. Decision Maker with a General Utility Function

An analogous expression can be derived for any utility function. In the general case, $U = (u_1, ..., u_n)$ denotes the utility gained in each of the n outcomes, i.e., $u_i = u(W + \pi_i)$. Similarly, $u_0 = u(W)$ denotes the utility of skipping the lottery.

Proposition 2 *For a decision maker with a strictly increasing utility function,*

$$\mathbb{E}\mathrm{U}_i = \begin{cases} |(u_i - u_0) \cdot p_i| & : \quad (\mathbb{E}\mathrm{U} - u_0) \cdot \pi_i < 0 \\ max\,(0, |(u_i - u_0) \cdot p_i| - |\mathbb{E}\mathrm{U} - u_0|) & : \quad (\mathbb{E}\mathrm{U} - u_0) \cdot \pi_i \geq 0 \end{cases}$$

Proof: As in Proposition 1, one can obtain an analogous expression for $\mathbb{E}\mathrm{U}_i$.

$$\mathbb{E}\mathrm{U}_i = \max\{u_0 \cdot p_i, u_i \cdot p_i\} + \max\{u_0 \cdot (1 - p_i), \mathbb{E}\mathrm{U} - u_i \cdot p_i\} - \max\{u_0, \mathbb{E}\mathrm{U}\} \quad (3.2)$$

Similarly, there are four cases to consider in 3.2

Case-1: If $\mathbb{E}\mathrm{U} < u_0$, $\pi_i > 0$, then $\mathbb{E}\mathrm{U}_i = p_i \cdot (u_i - u_0)$

Case-2: If $\mathbb{E}\mathrm{U} \leq u_0$, $\pi_i \leq 0$, then $\mathbb{E}\mathrm{U}_i = \max\{0, \mathbb{E}\mathrm{U} - p_i \cdot (u_i - u_0) - u_0\}$

Case-3: If $\mathbb{E}\mathrm{U} \geq u_0$, $\pi_i \geq 0$, then $\mathbb{E}\mathrm{U}_i = \max\{0, u_0 - p_i \cdot (u_0 - u_i) - \mathbb{E}\mathrm{U}\}$

Case-4: If $\mathbb{E}\mathrm{U} > u_0$, $\pi_i < 0$, then $\mathbb{E}\mathrm{U}_i = p_i \cdot (u_0 - u_i)$

One can combine these cases to obtain the final result. $\square$

Comparison of the value of specific information about a single state is possible with Proposition 2 for the general case where we have a risk averse decision maker. Obtaining information about a single state alters the optimal action on at most one possible occasion. The optimal action taken after assuming the information about outcome i is an $\mathcal{F}_i$-measurable function where $\mathcal{F}_i$ is the σ-algebra generated by E_i. With richer information bundles, it is possible to improve the expected gains from playing the lottery.

The decision maker is never worse off learning whether multiple events of the sort

E_i have occured or not. However, as the following proposition states, there are cases where obtaining finer information bundles does not improve the expected utility of the decision maker. We reach this conclusion by making a comparison between the information bundles generated by $E^m = \{\pi_1, ..., \pi_m\}$ and $\{\pi_1\}$,...,$\{\pi_m\}$ where $m \leq n$. Let $\mathcal{F}^m = \sigma(E^m)$ and $\hat{\mathcal{F}}^m = \sigma(E_1, ..., E_m)$. We denote the value of information conveyed by a σ-algebra, $\mathcal{F}$, by $\mathbb{E}\mathrm{U}[\mathcal{F}]$. Then

$$\mathbb{E}\mathrm{U}[\mathcal{F}^m] = (p_1 + ... + p_m) \cdot \max\,(u_0, (u_1 \cdot p_1 + ... + u_m \cdot p_m)/(p_1 + ... + p_m))+$$
$$(1 - p_1 - ... - p_m) \cdot \max\,(u_0, (\mathbb{E}U - u_1 \cdot p_1 - ... - u_m)/(1 - p_1 - ... - p_m)) - \max\,(u_0, \mathbb{E}\mathrm{U})$$
$$= \max\,(u_0 \cdot \{p_1 + ... + p_m\}, u_1 \cdot p_1 + ... + u_m \cdot p_m)+$$
$$\max\,(u_0 \cdot \{1 - p_1 - ... - p_m\}, \mathbb{E}\mathrm{U} - u_1 \cdot p_1 - ... - u_m) - \max\,(u_0, \mathbb{E}\mathrm{U})$$

Similarly,

$$\mathbb{E}\mathrm{U}[\hat{\mathcal{F}}^m] = p_1 \cdot \max\,(u_0, u_1) + ... + p_m \cdot \max\,(u_0, u_m)+$$
$$(1 - p_1 - ... - p_m) \cdot \max\,(u_0, (\mathbb{E}U - u_1 \cdot p_1 - ... - u_m)/(1 - p_1 - ... - p_m)) - \max\,(u_0, \mathbb{E}\mathrm{U}).$$

Hence, the improvement in the value of information from obtaining precise information about whether each outcome $i, 1 \leq i \leq m$ has occured or not is as follows,

$$\Delta\mathcal{F}^m = \mathbb{E}\mathrm{U}[\hat{\mathcal{F}}^m] - \mathbb{E}\mathrm{U}[\mathcal{F}^m] = p_1 \cdot \max\,(u_0, u_1) + ... + p_m \cdot \max\,(u_0, u_m) -$$
$$\max\,(u_0 \cdot \{p_1 + ... + p_m\}, u_1 \cdot p_1 + ... + u_m \cdot p_m).$$

Proposition 3

$$\Delta\mathcal{F}^m = \begin{cases} 0 & : \ \exists \ no \ i, j \leq m \ s.t. \ \pi_i \cdot \pi_j < 0 \\ > 0 & : \ o.w. \end{cases}$$

Proof: There are two cases to consider for the first equality. We will give a proof for the case where $\pi_i \geq 0$, and the proof for the case with $\pi_i < 0$ follows similarly. Since

$\pi_i \geq 0$, $u_i \geq u_0$ $\forall i$ s.t. $0 \leq i \leq m$. Then

$$\max\ (u_0 \cdot \{p_1 + ... + p_m\}, u_1 \cdot p_1 + ... + u_m \cdot p_m) = u_1 \cdot p_1 + ... + u_m \cdot p_m$$

$$p_1 \cdot \max\ (u_0, u_1) + ... + p_m \cdot \max\ (u_0, u_m) = u_1 \cdot p_1 + ... + u_m \cdot p_m$$

Hence, the result follows. Now, assume that $\exists\ i, j$ s.t. $\pi_i \cdot \pi_j < 0$ and wlog $\pi_i > 0$. Then $u_i > u_0$ and $u_j < u_0$. Hence

$$\begin{aligned}
\Delta\mathcal{F}^m &= u_i \cdot p_i + u_0 \cdot p_j + \textstyle\sum_{k=1,k\neq i,j}^m p_k \cdot \max\ (u_0, u_k) - \max\ (u_0 \cdot \sum_{k=1}^m p_k, \sum_{k=1}^m u_k \cdot p_k) \\
&\geq u_i \cdot p_i + u_0 \cdot p_j + \max\ (u_0 \cdot \textstyle\sum_{k=1,k\neq i,j}^m p_k, \sum_{k=1,k\neq i,j}^m p_k \cdot u_k) - \\
&\qquad \max(u_0 \cdot \textstyle\sum_{k=1}^m p_k, \sum_{k=1}^m u_k \cdot p_k) \\
&= \max\ (u_i \cdot p_i + u_0 \cdot p_j + u_0 \cdot \textstyle\sum_{k=1,k\neq i,j}^m p_k, u_i \cdot p_i + u_0 \cdot p_j + \sum_{k=1,k\neq i,j}^m p_k \cdot u_k) - \\
&\qquad \max\ (u_0 \cdot \textstyle\sum_{k=1}^m p_k, \sum_{k=1}^m u_k \cdot p_k) \\
&> \max\ (u_0 \cdot \textstyle\sum_{k=1}^m p_k, \sum_{k=1}^m u_k \cdot p_k) - \max\ (u_0 \cdot \sum_{k=1}^m p_k, \sum_{k=1}^m u_k \cdot p_k) = 0\ \square
\end{aligned}$$

III.1.3. Evaluation of Information on the Range of the Outcome

Suppose that we enumerate the n outcomes such that $\pi_i < \pi_j$ for $i < j$. In this section, we would like to compare information bundles generated by the events of the form $E^{\leq m} = \{\pi_1, ..., \pi_m\}$. In other words, the decision maker is allowed to ask the question, "Is the outcome less than or equal to π_m?". The value of information conveyed by such an event is given in the next proposition.

Proposition 4 *Let $\mathcal{F}^{\leq m}$ be the information bundle generated by $E^{\leq m}$. Then, for a decision maker with a general utility function*

$$\mathbb{E}\mathrm{U}[\mathcal{F}^{\leq m}] = \begin{cases} max\ (0, (\mathbb{E}\mathrm{U} - u_0) - \sum_{i\leq m}(u_i - u_0) \cdot p_i) & : \quad \mathbb{E}\mathrm{U} < u_0 \\ max\ (0, \sum_{i\leq m}(u_0 - u_i) \cdot p_i) & : \quad \mathbb{E}\mathrm{U} \geq u_0 \end{cases}$$

Proof: The computation proceeds as follows

$$\begin{aligned}
\mathbb{E}\mathrm{U}[\mathcal{F}^{\leq m}] &= (p_1 + .. + p_m) \cdot \max\ (u_0, (u_1 \cdot p_1 + ... + u_m \cdot p_m)/(p_1 + ... + p_m)) + \\
&\qquad (1 - p_1 - ... - p_m) \cdot \max\ (u_0, (\mathbb{E}U - u_1 \cdot p_1 - ... - u_m \cdot p_m)/(1 - p_1 - ... - p_m)) -
\end{aligned}$$

$$
\begin{aligned}
&\max\ (u_0, \mathbb{E}\mathrm{U}) \\
&= \max\ (u_0 \cdot (p_1 + ... + p_m), u_1 \cdot p_1 + ... + u_m \cdot p_m) + \\
&\quad \max\ (u_0 \cdot (1 - p_1 - ... - p_m), \mathbb{E}\mathrm{U} - u_1 \cdot p_1 - ... - u_m \cdot p_m) \\
&\quad - \max\ (u_0, \mathbb{E}\mathrm{U})
\end{aligned}
$$

Case 1: If $\mathbb{E}\mathrm{U} < u_0$,

$$
\begin{aligned}
\mathbb{E}\mathrm{U}[\mathcal{F}^{\leq m}] &= u_0 \cdot (p_1 + ... + p_m - 1) + \\
&\qquad \max\ (u_0 \cdot (1 - p_1 - ... - p_m), \mathbb{E}\mathrm{U} - u_1 \cdot p_1 - ... - u_m \cdot p_m) \\
&= \max\ (0, (\mathbb{E}\mathrm{U} - u_0) - (u_1 - u_0) \cdot p_1 - ... - (u_m - u_0) \cdot p_m)
\end{aligned}
$$

Case 2: If $\mathbb{E}\mathrm{U} \geq u_0$,

$$
\begin{aligned}
\mathbb{E}\mathrm{U}[\mathcal{F}^{\leq m}] &= \max\ (u_0 \cdot (p_1 + ... + p_m), u_1 \cdot p_1 + ... + u_m \cdot p_m) \\
&\qquad + \mathbb{E}\mathrm{U} - u_1 \cdot p_1 - ... - u_m \cdot p_m - \mathbb{E}\mathrm{U}. \\
&= \max\ ((u_0 - u_1) \cdot p_1 + ... + (u_0 - u_m) \cdot p_m, 0)
\end{aligned}
$$

□

The decision maker should ask the right question to maximize his expected utility before obtaining the information. The value of information is maximized at m that offers the maximum payoff among negative outcomes. This is shown in the following corollary.

Corollary 3 $\mathbb{E}\mathrm{U}[\mathcal{F}^{\leq m}]$ *is maximized at the value of* m^* *such that* $\pi_{m^*} = max\ \{\pi_i : \pi_i \leq 0$ *and* $\pi_{i+1} > 0\}$.

Proof: First, suppose that $\mathbb{E}\mathrm{U} < u_0$,

$$
\mathbb{E}\mathrm{U}[\mathcal{F}^{\leq m}] = \max\ (0, (\mathbb{E}\mathrm{U} - u_0) - \sum_{i \leq m} (u_i - u_0) \cdot p_i). \tag{3.3}
$$

Since for each i such that $\pi_i \leq 0$, $u_i \leq u_0$, 3.3 is maximized by involving all the negative outcomes. Similar observation for the opposite case leads to the desired conclusion. □

In this lottery setting, one does not have to know explicitly which outcomes have occured if the payoff in those states are all positive or negative. The action space of the decision maker involves only two actions which suggests in the light of Proposition 3 that he cares only about learning whether the payoff is positive or negative. Accordingly, a direct result of Proposition 3 is the existence of some information bundle generated by a single event that has exactly the same value as the information bundle which reveals perfect information about the states of the world before decision is made. Furthermore, this event does not have to induce a partition of the outcome space that is payoff adequate. The only payoff adequate partition of the outcome space is the information bundle that provides perfect information. However, obtaining perfect information may be a very costly activity. In this regard, it is important to realize that the decision maker does not have to obtain perfect information. We know that $EVPI = \sum_{\{\pi_i|\pi_i>0\}} p_i \cdot u_i + u_0 \cdot \sum_{\{\pi_i|\pi_i<0\}} p_i$.

Corollary 4 $\mathbb{E}\mathrm{U}[\mathcal{F}^{\leq m^*}] = EVPI$.

Proof: Let $\mathcal{F}^p = \sigma(\{\pi_i\}|\pi_i > 0)$. By Proposition 3, $\mathbb{E}\mathrm{U}[\mathcal{F}^p] = \mathbb{E}\mathrm{U}[\mathcal{F}^{\leq m^*}]$. It is clear that:

$$\mathbb{E}\mathrm{U}[\mathcal{F}^p] = \sum_{\{\pi_i|\pi_i>0\}} p_i \cdot u_i + u_0 \cdot \sum_{\{\pi_i|\pi_i<0\}} p_i.$$

which is equal to $EVPI$. $\square$

Hence, information about $E^{\leq m^*}$ does not only offer the maximum expected utility among the class of information bundles generated by events of the form $E^{\leq m}$, but also among the class of all possible information bundles. A similar result can obtained in the case of finite action and finite outcome space without any difficulty. To see this, let $A = \{a_1, ..., a_k\}$ be the finite action space and $u_{i,j}$ be the utility earned when π_i is observed and action a_j is taken. Next, define $A^i = \{\pi_s : u_{s,i} = \mathrm{argmax}_{a_j \in A} u_{s,j}\}$. Then

$$\mathbb{E}\mathrm{U}(\sigma(A^1, A^2, ..., A^k)) = \sum_{j=1}^{k} [\{\sum_{\pi_s \in A^j} \frac{p_s}{\sum_{\pi_t \in A^j} p_t} u_{s,j}\} \sum_{\pi_t \in A^j} p_t] - \mathbb{E}\mathrm{U}$$

$$= \sum_{j=1}^{k} \sum_{\pi_s \in A^j} p_s u_{s,j} - \mathbb{E}\mathrm{U}$$

$$= \sum_{\pi_s} p_s \ \mathrm{argmax}_{a_j \in A} u_{s,j} - \mathbb{E}\mathrm{U}$$

$$= EVPI$$

In the general case, it is not possible to find a single event that will yield the same value as the perfect information. The decision maker's objective should be to gather information about events on which a single action is optimal. Intuitively, there is no gains in learning the exact outcome in an event which induces a single action. If the decision maker asks the right questions, then he can avoid the costs of obtaining very precise information.

III.2. Risk Aversion and Ranking of Information

The decision maker's utility function is an important determinant of the value of information. The shape of the utility function describes the degree of risk aversion. As shown in Gould [12] and Hilton [14], there is no general monotonic relation between the value of information and risk taking behavior of the decision maker. This conclusion does not preclude any monotonic relation that holds for a particular class of decision makers or probability distributions. Ohlson [30] and Willinger [39] showed such a relation holds if the probability distribution is small risk in the sense of Samuelson [35].

We show below that there is a monotonic relation between the degree of risk aversion and the ranking of the information bundles $\mathcal{F}_i$, if we limit the class of utility functions. We consider two decision makers with strictly concave utility functions u and v. In what follows, v displays a lower degree of absolute risk aversion than u. The degree of absolute risk aversion is measured by $r_u(x) = -u''(x)/u'(x)$. We also let $\mathbb{E}$U and $\mathbb{E}$V denote the expected utility earned playing the lottery from decision makers with utility functions u and v respectively.

Proposition 5 *Let u and v be strictly concave and increasing utility functions with $r_v(x) \leq r_u(x)$ and π_i and π_j be two outcomes s.t. $\pi_i > \pi_j$. Suppose that u and v satisfy* *either one of the conditions below,*

(i) $(\mathbb{E}\mathrm{U} - u_0) \cdot \pi_i < 0$, $(\mathbb{E}\mathrm{U} - u_0) \cdot \pi_j < 0$, $(\mathbb{E}\mathrm{V} - v_0) \cdot \pi_i < 0$ *and* $(\mathbb{E}\mathrm{V} - v_0) \cdot \pi_j < 0$,

(ii) $(\mathbb{E}\mathrm{U} - u_0) \cdot \pi_i \geq 0$, $(\mathbb{E}\mathrm{U} - u_0) \cdot \pi_j \geq 0$, $(\mathbb{E}\mathrm{V} - v_0) \cdot \pi_i \geq 0$ *and* $(\mathbb{E}\mathrm{V} - v_0) \cdot \pi_j \geq 0$,

Then if u ranks information on outcome π_i above information on outcome π_j, v forms the same ranking between π_i and π_j.

Proof: Note that under both conditions, we have $\pi_i \cdot \pi_j > 0$. Under (i), we know from Proposition 2 that the information about outcome i has higher value than the information about outcome j if $|(u_i - u_0) \cdot p_i| > |(u_j - u_0) \cdot p_j|$. This condition can be rewritten as

$$\frac{u(W + \pi_i) - u(W)}{u(W + \pi_j) - u(W)} > \frac{p_i}{p_j}.$$

Since $r_v(x) \leq r_u(x)$, there exists an increasing concave function ψ such that $u(x) = \psi(v(x))$ (see [24]). By concavity and since $\pi_i > \pi_j$, we have

$$\frac{v(W + \pi_i) - v(W)}{v(W + \pi_j) - v(W)} \geq \frac{\psi(v(W + \pi_i)) - \psi(v(W))}{\psi(v(W + \pi_j)) - \psi(v(W))} > \frac{p_i}{p_j}.$$

The desired conclusion follows from here.

Now, under (ii), we retain the relation $\pi_i \cdot \pi_j > 0$. First, assume that both π_i and π_j are positive. Then $\mathbb{E}\mathrm{U} \geq u_0$ and $\mathbb{E}\mathrm{V} \geq v_0$ holds. By Proposition 2, the information bundle $\mathcal{F}_i$ is more valuable than $\mathcal{F}_j$ if $\max(0, (u_i - u_0) \cdot p_i - (\mathbb{E}\mathrm{U} - u_0)) \geq \max(0, (u_j - u_0) \cdot p_j - (\mathbb{E}\mathrm{U} - u_0))$. This condition is exactly the same as the condition in the first case, so the result follows. Finally, we assume that both π_i and π_j are negative. This clearly implies that $\mathbb{E}\mathrm{U} < u_0$ and $\mathbb{E}\mathrm{V} < v_0$, and using exactly the same condition as in the first assumption, we can show that the result follows. $\square$

This proposition derives the conditions under which we can make any comparison between the preference ranking of different information bundles among decision makers with various attitudes towards risk. However, the postulated assumptions are restrictive and the proposition provides comparison among a limited class of decision makers.

This is an indication of a lack of relation between the degree of risk aversion and the value of information even when we would like to compare a small class of information bundles. Intuitively, it is possible to form an opinion in favor of higher demand for information when the decision maker is more risk averse. However, we are considering the value of information *a priori*. Therefore, there is always some risk inherent in the process of gathering of information. As we have incorporated in our analysis, making the decision in the light of an information bundle may not induce a change in inital decision in many cases. It may even be true that the decision maker decides to change his initial action with knowledge about one low probability event only in case that one event with a low probability occurs. This scenario poses a risk of getting valueless information, hence the decision maker with a higher degree of risk aversion may value this information bundle very poorly *a priori* despite the fact that learning one of the events in the partition may save him from drastic consequences.

For a more restricted class of utility functions, it is possible to reach similar conclusions in other cases, too. We demonstrate this with an example.

III.2.1. An Example to Rank Information Bundles

Let's assume that the decision maker has the utility function $u_r(x) = -e^{-r \cdot x}$. A nice property of this family of utility functions is that r is the coefficient of absolute risk aversion. Suppose that there are two decision makers with utility functions u_a and u_b where $a \geq b$. Furthermore, consider the case where $(\mathbb{E}\mathrm{U} - u_0) \cdot \pi_i < 0$ and $(\mathbb{E}\mathrm{U} - u_0) \cdot \pi_j \geq 0$, i.e., both u_a and u_b satisfies these two conditions. Retaining the assumption that $\pi_i > \pi_j$, we know from Proposition 2 that $\mathcal{F}_i$ is valued more than $\mathcal{F}_j$ if $|(u_i - u_0) \cdot p_i| \geq \max\left(0, |(u_j - u_0) \cdot p_j| - |\mathbb{E}\mathrm{U} - u_0|\right)$. Note that, $\pi_i \cdot \pi_j < 0$ under the case we consider and this implies $\pi_i > 0$ and $\pi_j < 0$. Incorporating these, the condition becomes

$$\frac{u_i - u_0}{u_0 - u_j} \geq \frac{p_j}{p_i} - \frac{u_0 - \mathbb{E}U}{p_i \cdot (u_0 - u_j)}.$$

By the concavity of the utility functions, we may deduce that

$$\frac{u_b(W + \pi_i) - u_b(W)}{u_b(W) - u_b(W + \pi_j)} \geq \frac{u_a(W + \pi_i) - u_a(W)}{u_a(W) - u_a(W + \pi_j)}.$$

We need to see how $(u_0 - \mathbb{E}U)/(p_i \cdot (u_0 - u_j))$ behaves as the coefficient of absolute risk aversion, r, increases

$$\frac{u_0 - \mathbb{E}\mathrm{U}}{p_i \cdot (u_0 - u_j)} = \frac{-1 + \sum_{k=1}^{n} p_k \cdot exp(-r \cdot \pi_k)}{-1 + exp(-r \cdot \pi_j)} \tag{3.4}$$

and, we take the derivative of 3.4 with respect to r

$$\frac{d(\frac{u_0 - \mathbb{E}\mathrm{U}}{p_i \cdot (u_0 - u_j)})}{dr} = \frac{(-\sum_{k=1}^{n} \pi_k \cdot p_k \cdot exp(-r \cdot \pi_k)) \cdot (-1 + exp(-r \cdot \pi_j))}{(-1 + exp(-r \cdot \pi_j))^2} +$$

$$\frac{\pi_j \cdot exp(-r \cdot \pi_j) \cdot (-1 + \sum_{k=1}^{n} p_k \cdot exp(-r \cdot \pi_k))}{(-1 + exp(-r \cdot \pi_j))^2}. \tag{3.5}$$

From the conditions given, we obtain $-1 + \sum_{k=1}^{n} p_k \cdot \exp(-r \cdot \pi_k) \geq 0$ and $-1 + \exp(-r \cdot \pi_j) \geq 0$. Since $\pi_j \leq 0$, $\pi_j \cdot \exp(-r \cdot \pi_j) \leq 0$. The sign of 3.5 will be negative if $\sum_{k=1}^{n} \pi_k \cdot p_k \cdot \exp(-r \cdot \pi_k)$ is positive. This depends clearly on the nature of the lottery. However

$$\frac{d(\sum_{k=1}^{n} \pi_k \cdot p_k \cdot exp(-r \cdot \pi_k))}{dr} = -\sum_{k=1}^{n} \pi_k^2 \cdot p_k \cdot exp(-r \cdot \pi_k) < 0$$

Hence, this term is monotonically decreasing in r. Furthermore

$$\sum_{k=1}^{n} \pi_k \cdot p_k \cdot exp(-r \cdot \pi_k)|_{r=0} = \sum_{k=1}^{n} \pi_k \cdot p_k = E[\text{outcome from the lottery}].$$

Hence, if the expected outcome from the lottery is positive, we can arrive a similar conclusion as in Proposition 5 as long as $a, b \in [0, \bar{r}]$, where $\bar{r}$ satisfies

$$\sum_{k=1}^{n} \pi_k \cdot p_k \cdot exp(-r \cdot \pi_k)|_{r=\bar{r}} = 0.$$

Otherwise, we cannot arrive at a certain conclusion.

This example illustrates the difficulty in claiming certain conclusions between the degree of risk aversion and ranking of information. The nature of the lottery and the utility functions has a non monotonic impact on the preference rankings of decision makers. In the next chapter, we turn our attention to the relation between the degree of risk aversion and the value of information and show that even in a restricted class of information bundles the direction of relation changes under different circumstances.

III.3. Value of Information and the Level of Initial Wealth

Another determinant of the value of information is the level of initial wealth. The decision maker's attitude towards the lottery depends on the level of wealth. The risk attitude is a function of wealth as long as the utility function of the decision maker does not exhibit constant degree of risk aversion. Even though the decision maker is risk averse, for high levels of wealth, he may exhibit a behavior similar to a risk neutral decision maker. In this short section, we show that a monotonic relation between the level of initial wealth and the value of information exists for information bundles $\mathcal{F}_i$.

Proposition 6 *Let $u(x)$ be a concave utility function. Then if $(\mathbb{E}\mathrm{U} - u_0) \cdot \pi_i < 0$, $\mathbb{E}\mathrm{U}_i$ is non-increasing in the level of initial wealth, W.*

Proof: First consider $\pi_i > 0$. By Proposition 2, $\mathbb{E}\mathrm{U}_i = (u(W+\pi_i) - u(W)) \cdot p_i$. Then, $d\mathbb{E}U_i/dW = (u'(W + \pi_i) - u'(W)) \cdot p_i$, and the result holds as $u'(W + \pi_i) \leq u'(W)$. The opposite case, $\pi_i < 0$, follows in a similar manner. □

One final observation is that the value of information about an outcome is non-increasing with the level initial wealth, W, for utility functions with a constant coefficient of absolute risk aversion. We know from La Valle [17] that, such a utility function can be represented as $u(x) = -\exp(-ax)$ where a is the coefficient of absolute risk aversion. We state this result in the proposition below.

Proposition 7 *Let $u(x)$ be a concave utility function with a constant coefficient of absolute risk aversion. Then $\mathbb{E}\mathrm{U}_i$ is non-increasing in the level of initial wealth, W.*

Proof: When $(\mathbb{E}\mathrm{U} - u_0) \cdot \pi_i < 0$, the result follows from Proposition 6, so we treat the opposite case here. From Proposition 2, $\mathbb{E}\mathrm{U}_i = \max\{0, |(u_i - u_0)| \cdot pi - |\mathbb{E}\mathrm{U} - u_0|\}$. Then,

$$|(u_i - u_0)| \cdot pi - |\mathbb{E}\mathrm{U} - u_0| = \; p_i \cdot |exp(-a \cdot W) - exp(-a \cdot (W + \pi_i))| - |\sum_k p_k \cdot (exp(-a \cdot W) - exp(-a \cdot (W + \pi_k))| \quad (3.6)$$

Since $d(|(u_i - u_0)| \cdot pi)/dW \leq 0$ (see proof of Proposition 6), we will finish the proof if we show that the derivative of the second piece in 3.6 is non-negative. When $\mathbb{E}\mathrm{U} \geq u_0$, $d(|\mathbb{E}U - u_0|)/dW = a \cdot exp(-a \cdot W)(\sum_k p_k \cdot exp(-a \cdot \pi_k) - 1)$. Since the coefficient of absolute risk aversion is constant, $(\sum_k p_k \cdot exp(-a \cdot \pi_k) - 1) > 0$ and this leads to the desired conclusion. The case of $\mathbb{E}\mathrm{U} < u_0$, follows similarly. $\square$

III.4. Numerical Results

The value of information about a particular outcome depends on the utility function, the level of outcome, the probability of the outcome and the expected payoff from the lottery. In this section, we analyze numerically the behavior of the value of information with respect to these determinants. We will differentiate between a risk neutral and risk averse decision maker and observe how the risk taking behavior of the decision maker impacts the value of information.

III.4.1. Risk Neutral Decision Maker

First, we consider a risk neutral decision maker with $u(x) = x$. In the risk neutral case, if the lottery is attractive when there is no information, then information about the positive outcomes with either a low probability or a low value will not be desirable.

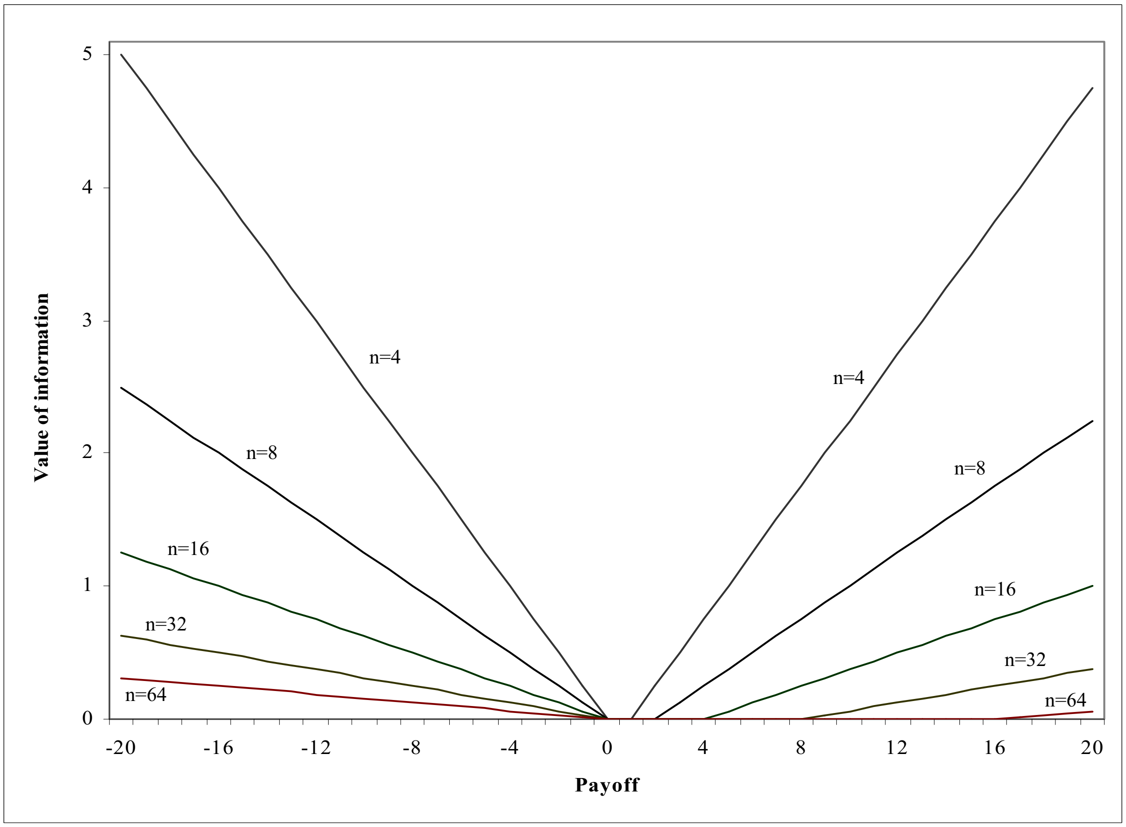

Fig. 1. Value of Information as the Number of Outcomes Increases.

If such an outcome is realized, the decision maker's action does not change. Similarly, if the outcome is not realized, the decision to play or not is unchanged, since the conditional expected payoff from the lottery is still positive. Given that such a outcome does not occur, the player still expects to earn a positive payoff. However, if an outcome with a low value has a higher likelihood relative to rest of the states, then the corresponding information may be valuable. The decision maker could be willing to play the lottery in no information case just because of the likelihood that this particular outcome will occur. In such a case, learning that particular outcome does not occur may induce the decision maker to skip the lottery instead.

In order to illustrate the effect of change in likelihood on the value of information, we imagine a lottery with expected payoff of \$0.25, in which all the outcomes are

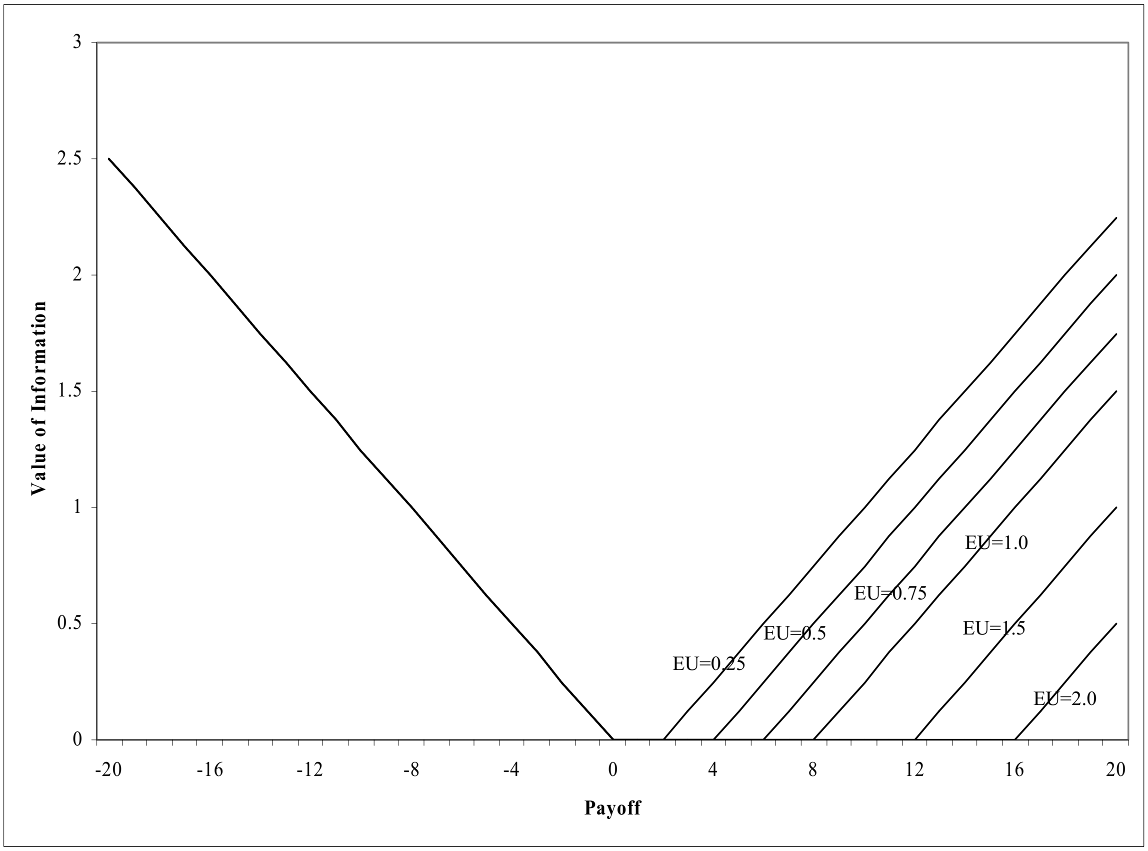

Fig. 2. Value of Information as Expected Payoff from the Lottery Increases when the Original Decision is to Play the Lottery.

equally likely. Under this setting, the number of possible outcomes have an effect on the value of information as the likelihoods of outcomes are influenced. The value of information about a single outcome does not depend on what the other outcomes are as long as the expected payoff remains the same. As the number of outcomes increases, the likelihood of each outcome decreases, and value of information about an outcome decreases. The impact of information on the original decision diminishes as the likelihood of this particular outcome approaches zero. Figure 1 illustrates this.

Figure 2 shows the behavior in different lotteries having different expected payoffs. All the expected payoffs from different lotteries were taken as positive. The value of information about positive outcomes decreases linearly. Learning such outcomes occur

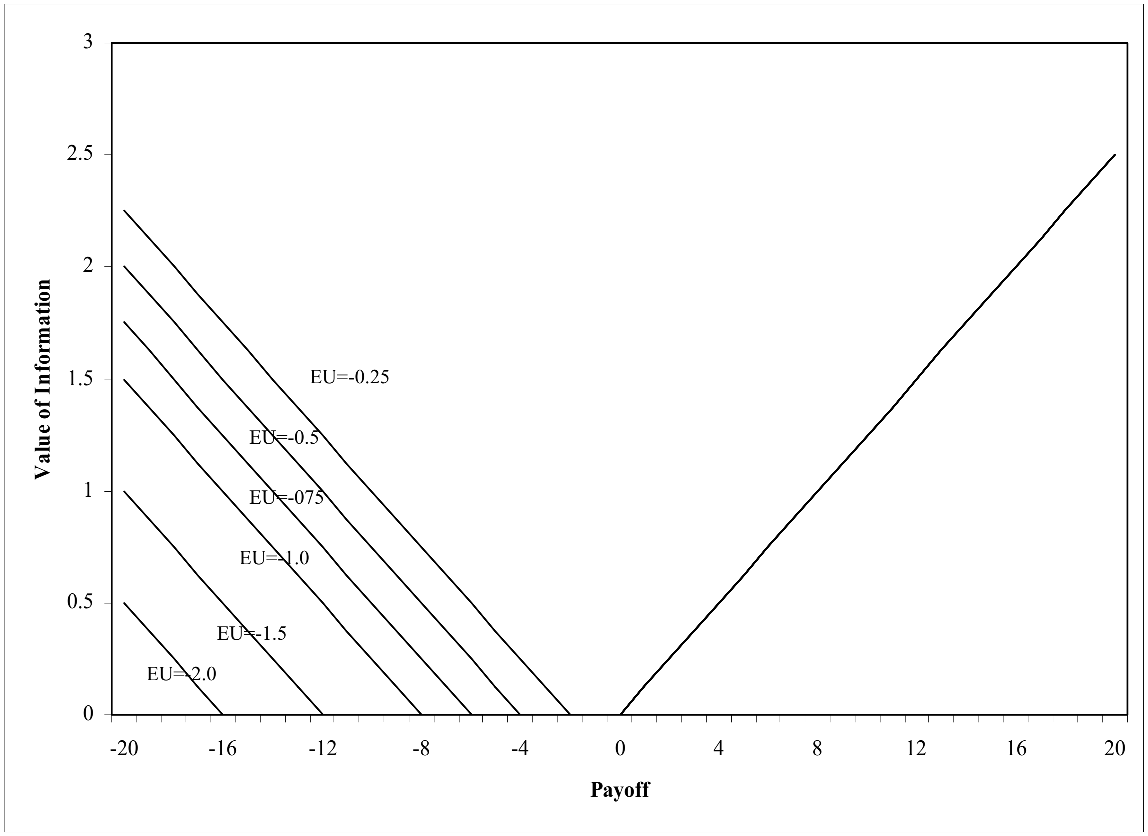

Fig. 3. Value of Information as Expected Payoff from the Lottery Increases when the Original Decision is to Skip the Lottery.

do not reveal any valuable information, however some valuable information might be revealed when they don't occur. As the expected payoff of the original lottery increases, they have less impact on the sign of the conditional expected payoff given they don't occur. Therefore, the value of information decreases linearly. Similar behavior is not observed for the negative outcomes. The expected payoff earned from the lottery has no impact on the value of information. When the decision maker learns that a negative outcome is not realized, there is no reason to change the original decision. On the contrary, when such an outcome is realized, we know that the decision maker will make use of this piece of information to modify the decision. Thus, the expected value of the lottery has no impact on the value of information regarding negative outcomes.

Table 1. Value of Information on Different Outcomes as a Function of “n”.

Payoff	VOI for n=2	VOI for n=4	VOI for n=8	VOI for n=16
-15	1620009.186	802755.0931	394128.0466	189814.5233
-13	206707.196	96104.098	40802.549	13151.7745
-11	15437.57086	469.2854288	0	0
-10...0	0	0	0	0
1	0.316060279	0.15803014	0.07901507	0.039507535
2	0.432332358	0.216166179	0.10808309	0.054041545
3	0.475106466	0.237553233	0.118776616	0.059388308
4	0.490842181	0.24542109	0.122710545	0.061355273
10	0.4999773	0.24998865	0.124994325	0.062497163
15	0.499999847	0.249999924	0.124999962	0.062499981

However, the opposite conclusion prevails when the numerical results are determined assuming negative expected payoff from the original lottery. Figure 3 illustrates this second case.

III.4.2. Risk Averse Decision Maker

The more interesting case is a risk averse decision maker. We will consider a utility function with constant coeffient of absolute risk aversion, $u(x) = -exp(-ax)$. The decision maker becomes more risk averse as a increases. As the decision maker becomes more risk averse, the certainty equivalent of the lottery increases. We will determine the value of information regarding a particular outcome by considering an imaginary lottery in which outcomes are equally likely. The decision maker has an initial level of wealth W. Table 1 illustrates the value of information for different outcomes in a lottery which yields the decision maker an expected utility of -14500. The coefficient of risk aversion is taken as 1. This choice of expected utility quantity might seem unusual at first, however, the particular utility function is an exponential function. As opposed to the risk neutral case, the initial decision is to skip the lottery. Hence, any positive outcome conveys valuable information. However, the value of information regarding a positive outcome is quite insignificant. This is mainly due to the particular form of the exponential utility function. Marginal utility earned from increasing the outcome diminishes rapidly. As n increases, the value of information decreases for each outcome. The positive outcomes convey valuable information when they occur, so the decrease in their likelihood decreases the possibility of obtaining valuable information. Conversely, the negative outcomes can only make an impact when they do not occur. However, as a negative outcome becomes less likely, the decision maker may not be better off learning this particular outcome is not realized because he may still expect to do worse.

As in the risk neutral case, the value of information about an outcome depends on the rest of the outcomes only through the expected utility earned by playing the lottery. As long as the expected utility remains below the utility earned by skipping the lottery, the value of information regarding positive outcomes remains the same. The same behavior is observed for negative outcomes as long as the expected utility is higher than the utility earned from skipping the lottery. In those case, the value of information depends only on the likelihood of the particular outcome. Table 2 illustrates the value of information about several outcomes in different imaginary lotteries as the expected utility earned from the lottery changes. We fixed $n = 8$ and $a = 1$. It is not surprising to observe that the value of information regarding positive outcomes remains the same. The reason behind this is exactly the same as in risk neutral case: the value of information regarding a positive outcome comes from modifying the decision when

Table 2. Value of Information for Different Outcomes as a Function of Expected Utility Earned without Information.

Payoff	VOI EU=-100000	VOI EU=-10000	VOI EU=-1000	VOI EU=-100	VOI EU=-1
-15	308628.0466	398628.0466	407628.0466	408528.0466	408627.046559
-13	0	45302.549	54302.549	55202.549	55301.549001
-11	0	0	6485.142714	7385.142714	7484.142714
-10	0	0	1754.183224	2654.183224	2753.183224
-7	0	0	0	37.9541448	136.954145
-3	0	0	0	0	2.385692
-1	0	0	0	0	0.214785
1	0.07901507	0.07901507	0.07901507	0.07901507	0.07901507
2	0.10808309	0.10808309	0.10808309	0.10808309	0.10808309
5	0.124157757	0.124157757	0.124157757	0.124157757	0.124157757
10	0.124994325	0.124994325	0.124994325	0.124994325	0.124994325
15	0.124999962	0.124999962	0.124999962	0.124999962	0.124999962

that outcome is realized, so the expected utility earned from playing the lottery has no impact on this. On the other hand, the value of information regarding negative outcomes increases as the expected utility earned from the lottery gets closer to the utility earned from skipping the lottery. The player considers a change in action when such an outcome is not realized as the initial decision becomes less clear cut.

The level of risk aversion also has an impact on the value of information, although there is no monotonic relation between two. Suppose $(\mathbb{E}\text{U}-u_0)\cdot\pi_i < 0$ holds. Then from Proposition 2, $\mathbb{E}\text{U}_i = |(u_i - u_0)\cdot p_i|$. When $\pi_i > 0$, $\mathbb{E}\text{U}_i = -p_i.exp(-aW)(exp(-a\pi_i) -$

Table 3. Value of Information for Different Outcomes as a Changes When $W = 0$.

Payoff	VOI for a=0.1	VOI for a=0.5	VOI for a=0.8	VOI for a=1	VOI for a=2
-15	0	0	5845.223927	394128.0466	$1.3358\ 10^{12}$
-13	0	0	0	40802.549	$2.4466\ 10^{10}$
-11	0	0	0	0	$4.481\ 10^{8}$
-7	0	0	0	0	$1.3583\ 10^{5}$
-3	0	0	0	0	0
-1	0	0	0	0	0
1	0.011895	0.049184	0.068834	0.079015	0.108083
2	0.022659	0.079015	0.099763	0.108083	0.122711
5	0.049184	0.114739	0.122711	0.124158	0.124994
10	0.079015	0.124158	0.124958	0.124994	0.125000
15	0.097109	0.124931	0.124999	0.125000	0.125000

1), and $\frac{d\mathbb{E}\mathrm{U}_i}{da} = p_i \cdot exp(-aW) \cdot (exp(-a\pi_i)(W + \pi_i) - W)$. Clearly, it is not possible to sign this expression. Similar observation can be made for the other cases, so it is not possible to find a monotonic relationship between the value of information and the degree of absolute risk aversion. In numerical examples, the expected utility from the original lottery is taken as -14500 and the number of possible outcomes are fixed at $n = 8$. First, the level of initial wealth, W, is fixed at 0. As observed in Table 3, the value of information increases with coefficient of absolute risk aversion. In this table, the coefficient of absolute risk aversion does not have an impact on the maximum value of information regarding a positive outcome, which is 0.12500. However, as a increases, the convergence is much faster. Note that when $n = 8$, the value of information cannot

Table 4. Value of Information for Different Outcomes as a Changes When $W = 10$.

Payoff	VOI for a=0.1	VOI for a=0.5	VOI for a=0.8	VOI for a=1	VOI for a=2
-15	0	0	0	0	0
-13	0	0	0	0	0
-11	0	0	0	0	0
-7	0	0	0	0	0
-3	0	0	0	0	0
-1	0	0	0	0	0
1	0.004376	0.000331	$2.3091\ 10^{-5}$	$3.5873\ 10^{-6}$	$2.2278\ 10^{-10}$
2	0.008336	0.000532	$3.3467\ 10^{-5}$	$4.907\ 10^{-6}$	$2.5293\ 10^{-10}$
5	0.018094	0.000773	$4.1165\ 10^{-5}$	$5.6367\ 10^{-6}$	$2.5763\ 10^{-10}$
10	0.029068	0.000837	$4.1919\ 10^{-5}$	$5.6747\ 10^{-6}$	$2.5764\ 10^{-10}$
15	0.035724	0.000842	$4.1933\ 10^{-5}$	$5.675\ 10^{-6}$	$2.5764\ 10^{-10}$

exceed $\frac{1}{8}(-\exp(-a\pi_i) + 1)$ and the higher the value of a, the faster the utility function converges to zero. For the negative outcome, as the decision maker gets more risk averse, the impact of learning that such an outcome is not realized is less on the initial decision. The player is less willing to reconsider the initial decision; he has to make sure that a negative outcome is not realized to change his mind. Second, the level of initial wealth is fixed at 10, and one can observe that the value of information decreases with the coefficient of absolute risk aversion. This is illustrated in Table 4.

CHAPTER IV

BUYING PRICE OF INFORMATION IN LOTTERIES

Buying price, $b_{\mathcal{F},u}$, of information bundle $\mathcal{F}$ provide a nice way to compare value of information among different individuals as it is expressed in monetary terms. As in Chapter III, the decision maker faces the same decision of either playing or skipping a lottery with a finite number possible outcomes. Hence, the definition of the buying price slightly different than in Definition 7, where the decision maker is assumed to keep his initial wealth without playing a lottery in case no information is acquired. In the general case, the decision maker may choose an action from the set $\mathcal{A}$ even when he does not purchase information. Thus, $b_{\mathcal{F},u}$ is defined as follows.

Definition 9 *The buying price of information bundle $\mathcal{F}$ for a decision maker with a utility function $u(p(\pi,a))$ is:*

$$\mathbb{E}[\max_{a\in\mathcal{A}_{\mathcal{F}}}\mathbb{E}[u(p(\pi,a)-b_{\mathcal{F},u})|\mathcal{F}]] = \max_{a\in\mathcal{A}}\mathbb{E}[u(p(\pi,a))] \tag{4.1}$$

where $p(\pi,a)$ is the payoff when the outcome of the lottery is π and the decision maker takes action a, $\mathcal{A}$ is the set of available actions when no information is obtained and $\mathcal{A}_{\mathcal{F}}$ is the set of available actions when information bundle $\mathcal{F}$ is purchased.

The buying price of the information bundle is the maximum price the decision maker would be willing to pay to purchase the information. In this chapter, we continue our discussion to compare the value of information bundles $\mathcal{F}_i$. Let $b_{i,u}$ denote the buying price of $\mathcal{F}_i$ for a decision maker with a utility function u. We will determine the cases where it is possible to draw certain conclusions about the relation between risk aversion and the value of information.

IV.1. Buying Price vs Expected Utility Approach

As Hazen and Sounderpian [13] shows, the preference ranking of information is not the same under both approaches. In this section, we address the relation between two approaches, and show when the preference reversals occur. In this discussion, we restrict ourselves to the information bundles $\mathcal{F}_i$.

Proposition 8 *Let $u(x)$ be a strictly increasing utility function and let π_i be an outcome of the lottery.* $\mathbb{E}\mathrm{U}_i > 0 \Leftrightarrow b_{i,u} > 0$.

Proof: We start with the case $(\mathbb{E}\mathrm{U} - u_0) \cdot \pi_i \leq 0$. First assume that $\pi_i > 0$ and $u(W) \geq \mathbb{E}\mathrm{U}$. Then $b_{i,u}$ satisfies

$$u(W) = p_i \cdot u(W+\pi_i-b_{i,u}) + (1-p_i) \cdot \max\{u(W-b_{i,u}), \sum_{k\neq i} \frac{p_k}{1-p_i} \cdot u(W+\pi_k-b_{i,u})\} \quad (4.2)$$

Assume that $b_{i,u} = 0$. Then the right hand side of the equation 4.2 reduces to $\max\{p_i \cdot u(W+\pi_i) + (1-p_i) \cdot u(W), \sum_{k=1}^{n} p_k \cdot u(W+\pi_k)\}$. Since $\pi_i > 0$, the certainty equivalent of the lottery $\{\pi_k, p_k/(1-p_i)\}$ is less than W because the utility function $u(x)$ is strictly increasing. Hence $p_i \cdot u(W+\pi_i) + (1-p_i) \cdot u(W) > \sum_{k=1}^{n} p_k \cdot u(W+\pi_k)$. However, $p_i \cdot u(W+\pi_i) + (1-p_i) \cdot u(W) > u(W)$, which contradicts 4.2 when $b_{i,u} = 0$. Since the right hand side of 4.2 is continuous and decreasing in $b_{i,u}$, $\exists\, b_{i,u} > 0$ that satisfies the equation. The converse implication follows from Proposition 2.

When $\pi_i < 0$ and $u(W) \leq \mathbb{E}\mathrm{U}$, $b_{i,u}$ satisfies

$$\mathbb{E}\mathrm{U} = p_i \cdot u(W-b_{i,u}) + (1-p_i) \cdot \max\{u(W-b_{i,u}), \sum_{k\neq i} \frac{p_k}{1-p_i} \cdot u(W+\pi_k-b_{i,u})\} \quad (4.3)$$

Assume again that $b_{i,u} = 0$. Then the right hand side in 4.3 becomes $\max\{u(W), p_i \cdot u(W) + \sum_{k\neq i} p_k \cdot u(W+\pi_k)\}$. Since $\pi_i < 0$ and $u(x)$ is strictly increasing, $u(W) < p_i \cdot u(W) + \sum_{k\neq i} p_k \cdot u(W+\pi_k)$. However, $\mathbb{E}\mathrm{U} < p_i \cdot u(W) + \sum_{k\neq i} p_k \cdot u(W+\pi_k)$, which is a contradiction. As in the first case, $\exists\, b_{i,u} > 0$ that satisfies the equation. The converse implication follows from Proposition 2

Next, consider the case $(\mathbb{E}\mathrm{U} - u_0) \cdot \pi_i > 0$. When $\pi_i > 0$, $u(W) \leq \mathbb{E}\mathrm{U}$. Then $\mathbb{E}\mathrm{U}_i \geq 0$ implies that $(u(W + \pi_i) - u(W)) \cdot p_i > \mathbb{E}\mathrm{U} - u(W)$. After a little algebra, we obtain $u(W) > \sum_{k \neq i} (p_k/(1 - p_i)) \cdot u(W + \pi_k)$. The equation for $b_{i,u}$ is

$$\mathbb{E}\mathrm{U} = \max\{p_i \cdot u(W + \pi_i - b_{i,u}) + (1 - p_i) \cdot u(W - b_{i,u}), \sum_k p_k \cdot u(W + \pi_k - b_{i,u})\} \quad (4.4)$$

If $b_{i,u} = 0$ is the solution for 4.4, the equation becomes

$$\sum_k p_k \cdot u(W + \pi_k) = \max\{p_i \cdot u(W + \pi_i) + (1 - p_i) \cdot u(W), \sum_k p_k \cdot u(W + \pi_k)\}$$

which implies $u(W) \leq \sum_{k \neq i} (p_k/(1 - p_i)) \cdot u(W + \pi_k)$ after a little algebra. This is a contradiction. Hence, $b_{i,u} > 0$. Conversely, assume that $b_{i,u} > 0$. Then, $\sum_k p_k \cdot u(W + \pi_k) = p_i \cdot u(W + \pi_i - b_{i,u}) + (1 - p_i) \cdot u(W - b_{i,u})$. Since the right hand side of this inequality is strictly decreasing in $b_{i,u}$, $\sum_k p_k \cdot u(W + \pi_k) < p_i \cdot u(W + \pi_i) + (1 - p_i) \cdot u(W)$, or after a little arrangement we obtain $(u(W + \pi_i) - u(W)) \cdot p_i > \mathbb{E}\mathrm{U} - u(W)$, which is the condition for $\mathbb{E}\mathrm{U}_i > 0$.

The last case we consider is $\pi_i < 0$ and $\mathbb{E}\mathrm{U} \leq u(W)$. Assume first that $\mathbb{E}\mathrm{U}_i > 0$. This implies $\sum_{k \neq i} p_k \cdot u(W + \pi_k) > u(W) \cdot (1 - p_i)$. If $b_{i,u} = 0$, then $u(W) \geq p_i \cdot u(W) + \sum_{k \neq i} p_k \cdot u(W + \pi_k)$, which is a contradiction. Conversely, assume that $b_{i,u} > 0$, then $u(W) = p_i \cdot u(W - b_{i,u}) + \sum_{k \neq i} p_k \cdot u(W + \pi_k - b_{i,u})$. Since u is strictly decreasing in $b_{i,u}$, this equality implies $(1 - p_i) \cdot u(W) < \sum_{k \neq i} p_k \cdot u(W + \pi_k - b_{i,u})$, so $\mathbb{E}\mathrm{U}_i > 0$. Hence, under all possible cases $\mathbb{E}\mathrm{U}_i > 0 \Leftrightarrow b_{i,u} > 0$. $\square$

Corollary 5 *Let $u(x)$ be a strictly increasing utility function and let π_i be an outcome of the lottery. Then $(\mathbb{E}\mathrm{U} - u_0) \cdot \pi_i \leq 0$ implies that $b_{i,u} > 0$.*

Proof: By Proposition 2 $(\mathbb{E}\mathrm{U} - u_0) \cdot \pi_i \leq 0 \Rightarrow \mathbb{E}\mathrm{U}_i > 0$ and the result follows from Proposition 8. $\square$

Proposition 8 and Corollary 5 state a result that is analogous to Proposition 2. The information always has positive value when there is a possibility of changing the initial decision. Proposition 8 and Corollary 5 say that decision maker pays a strictly positive amount for such information. It does not make any sense to pay for information otherwise because no decision maker has the motivation to pay for an information bundle that does not change the initial decision.

Corollary 6 *Let $u(x)$ be a strictly increasing utility function and let π_i be an outcome of the lottery. Then $\mathbb{E}\mathrm{U}_i = 0$ if and only if $b_{i,u} = 0$.*

Proof: Follows directly from Proposition 8. $\square$

The following result clarifies when the preference rankings are preserved and when the reversals occur for information bundles $\mathcal{F}_i$:

Proposition 9 *Let $u(x)$ be a strictly increasing and strictly concave utility function exhibiting non-increasing degree of risk aversion, and let $\pi_i > \pi_j, \pi_i, \pi_j \neq 0$ be two outcomes of the lottery. Then,*

(a) If $\pi_i \cdot \pi_j > 0$, then $\mathbb{E}\mathrm{U}_j \geq \mathbb{E}\mathrm{U}_i$ implies $b_{j,u} \geq b_{i,u}$.

(b) If $\pi_i \cdot \pi_j < 0$, then $\mathbb{E}\mathrm{U}_i \geq \mathbb{E}\mathrm{U}_j$ implies $b_{i,u} \geq b_{j,u}$.

Proof: To prove (a), we consider two cases,

Case 1 ($0 > \pi_i > \pi_j$): Assume $\mathbb{E}\mathrm{U}_j \geq \mathbb{E}\mathrm{U}_i$. The desired conclusion holds whenever $\mathbb{E}\mathrm{U}_j = 0$ or $\mathbb{E}\mathrm{U}_i = 0$ by Corollary 6. Hence, we will only analyze the cases $\mathbb{E}\mathrm{U}_j, \mathbb{E}\mathrm{U}_i > 0$. Then $\mathbb{E}\mathrm{U}_j \geq \mathbb{E}\mathrm{U}_i$ implies $(u(W) - u(W + \pi_i)) \cdot p_i \leq (u(W) - u(W + \pi_j)) \cdot p_j$. Since $b_{i,u}, b_{j,u} > 0$, the equations for the buying prices are

$$\max\{\mathbb{E}\mathrm{U}, u(W)\} = p_i \cdot u(W - b_{i,u}) + \sum_{k \neq i} p_k \cdot u(W + \pi_k - b_{i,u}) \tag{4.5}$$

$$\max\{\mathbb{E}\mathrm{U}, u(W)\} = p_j \cdot u(W - b_{j,u}) + \sum_{k \neq j} p_k \cdot u(W + \pi_k - b_{j,u}) \tag{4.6}$$

Combining 4.5 and 4.6, we obtain

$$p_i \cdot u(W - b_{i,u}) + p_j \cdot u(W + \pi_j - b_{i,u}) + \textstyle\sum_{k \neq i,j} p_k \cdot u(W + \pi_k - b_{i,u}) = \\ p_j \cdot u(W - b_{j,u}) + p_i \cdot u(W + \pi_i - b_{j,u}) + \textstyle\sum_{k \neq i,j} p_k \cdot u(W + \pi_k - b_{j,u}) \quad (4.7)$$

Assume that $b_{i,u} > b_{j,u}$, then clearly $\sum_{k \neq i,j} p_k \cdot u(W + \pi_k - b_{i,u}) < \sum_{k \neq i,j} p_k \cdot u(W + \pi_k - b_{j,u})$. Then, if the equality holds, $p_i \cdot u(W - b_{i,u}) + p_j \cdot u(W + \pi_j - b_{i,u}) > p_j \cdot u(W - b_{j,u}) + p_i \cdot u(W + \pi_i - b_{j,u})$. Since $b_{i,u} > b_{j,u}$, the final inequality implies $p_i \cdot u(W - b_{j,u}) + p_j \cdot u(W + \pi_j - b_{j,u}) > p_j \cdot u(W - b_{j,u}) + p_i \cdot u(W + \pi_i - b_{j,u})$. After a little algebra

$$\frac{u(W - b_{j,u}) - u(W + \pi_j - b_{j,u})}{u(W - b_{j,u}) - u(W + \pi_i - b_{j,u})} < \frac{p_i}{p_j} \quad (4.8)$$

Since $u(x)$ exhibits non-increasing degree of risk aversion and it is strictly concave

$$\frac{u(W - b_{j,u}) - u(W + \pi_j - b_{j,u})}{u(W - b_{j,u}) - u(W + \pi_i - b_{j,u})} > \frac{u(W) - u(W + \pi_j)}{u(W) - u(W + \pi_i)} \geq \frac{p_i}{p_j} \quad (4.9)$$

Note that 4.9 follows from $\mathbb{E}\mathrm{U}_j \geq \mathbb{E}\mathrm{U}_i$. Clearly, 4.9 contradicts 4.8, so $b_{j,u} \geq b_{i,u}$.

Case 2 ($\pi_i > \pi_j > 0$): In this case, $\mathbb{E}\mathrm{U}_j \geq \mathbb{E}\mathrm{U}_i$ implies $(u(W + \pi_j) - u(W)) \cdot p_j \geq (u(W + \pi_i) - u(W)) \cdot p_i$. The buying prices, $b_{i,u}, b_{j,u} > 0$, satisfy

$$p_i \cdot u(W + \pi_i - b_{i,u}) + (1 - p_i) \cdot u(W - b_{i,u}) = p_j \cdot u(W + \pi_j - b_{j,u}) + (1 - p_j) \cdot u(W - b_{j,u}) \quad (4.10)$$

Assume $b_{i,u} > b_{j,u}$. Then, since $u(x)$ is strictly increasing, 4.10 implies

$$p_i \cdot u(W + \pi_i - b_{i,u}) + (1 - p_i) \cdot u(W - b_{i,u}) > p_j \cdot u(W + \pi_j - b_{i,u}) + (1 - p_j) \cdot u(W - b_{i,u}) \quad (4.11)$$

and after rearranging 4.11, we obtain

$$\frac{u(W + \pi_j - b_{i,u}) - u(W - b_{i,u})}{u(W + \pi_i - b_{i,u}) - u(W - b_{i,u})} < \frac{p_i}{p_j} \quad (4.12)$$

Inequality 4.12 implies a contradiction as in case 1, so $\mathbb{E}\mathrm{U}_j \geq \mathbb{E}\mathrm{U}_i$ implies $b_{j,u} \geq b_{i,u}$. Therefore, (a) holds.

Now, we prove (b). In this case, $\pi_i > 0 > \pi_j$. $\mathbb{E}\mathrm{U}_i \geq \mathbb{E}U_j$ implies $((u(W+\pi_i) - u(W)) \cdot p_i - (\mathbb{E}\mathrm{U} - u(W)) \geq (u(W) - u(W+\pi_j)) \cdot p_j$. After a little algebra, this becomes $u(W) \geq \sum_{k \neq i,j} \ (p_k/(1-p_i-p_j)) \cdot u(W+\pi_k)$. Then $b_{j,u}, b_{i,u} > 0$ satisfy the following equation

$$p_i \cdot u(W+\pi_i - b_{i,u}) + p_j \cdot u(W - b_{i,u}) + (1-p_i-p_j) \cdot u(W - b_{i,u}) =$$
$$p_i \cdot u(W+\pi_i - b_{j,u}) + p_j \cdot u(W - b_{j,u}) + (1-p_i-p_j) \cdot \textstyle\sum_{k \neq i,j} \ (p_k/(1-p_i-p_j)) \cdot u(W+\pi_k - b_{j,u})$$

If $b_{j,u} > b_{i,u}$, then $u(W - b_{i,u}) < \sum_{k \neq i,j} \ (p_k/(1-p_i-p_j)) \cdot u(W+\pi_k - b_{j,u})$ should hold, which is a contradiction as $u(x)$ exhibits non-increasing degree of risk aversion. Hence, $\mathbb{E}\mathrm{U}_i \geq \mathbb{E}\mathrm{U}_j$ implies $b_{i,u} \geq b_{j,u}$. □

Let $\mathcal{S}$ be the space of utility functions and the finite outcome lotteries including pairs (u, L) such that u is strictly concave and increasing, and L is simple. Both approaches of quantifying information partitions this space into two subsets in comparing the value of $\mathcal{F}_i$ and $\mathcal{F}_j$ at a given wealth W. Then Proposition 9 states that the relation between these two partitions can pictured as in Figure 4 when $\pi_i \cdot \pi_j > 0$. Note that, the partition is different when $\pi_i \cdot \pi_j < 0$. When the decision maker is asked to pay for acquisition of an information bundle, the nature of the lottery decision changes. If he actually agrees to pay, then the lottery decision is taken after the payment is made, i.e. at a lower level of wealth. This may result in a preference reversal on the information bundles. In the next example, we will illustrate cases where the preference reversals occur.

IV.1.1. An Illustration of Preference Reversals

We illustrate three cases where preference reversals occur. Consider a decision maker with the following utility function,

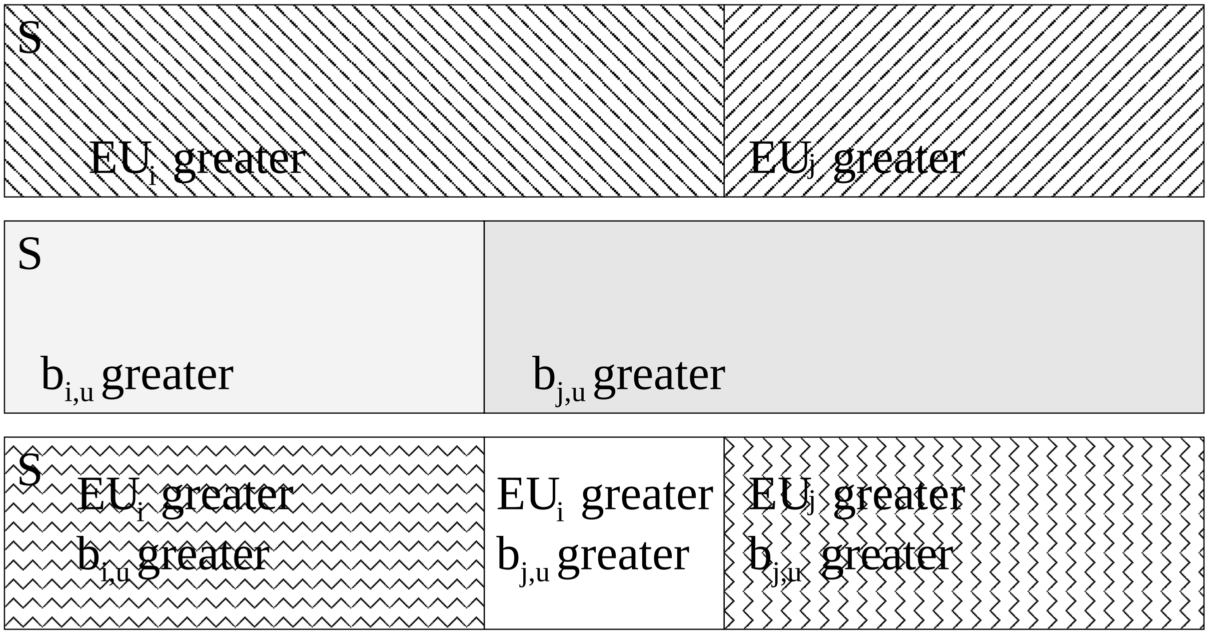

Fig. 4. Partition of $\mathcal{S}$ by Two Different Approaches of Quantifying Information when $\pi_i \cdot \pi_j > 0$.

Table 5. First Lottery in the Example.

Probability	0.125	0.125	0.125	0.125	0.175	0.075	0.125	0.125
Outcome, \$	10	3	-3	12	-4	-5	20	5

$$u(x) = \begin{cases} \sqrt{x+1} - 1 & : \quad x \geq 0 \\ \frac{1}{16} - (x - \frac{1}{4})^2 & : \quad x < 0 \end{cases}$$

This utility function is continuously differentiable and strictly concave. First, consider the lottery described in Table 5. If we compute the the value of information on outcomes -4 and -5, we obtain $\mathbb{E}\mathrm{U}_{-4} = 0.216312$ and $\mathbb{E}\mathrm{U}_{-5} = 0.205205$. However, $b_{-4,u} = 0.537605754$ and $b_{-5,u} = 0.596452717$. This illustrates a preference reversal when $\pi_i < 0, \pi_j < 0$ and $\pi_i > \pi_j$ in that $\mathbb{E}\mathrm{U}_i > \mathbb{E}\mathrm{U}_j$ does not necessarily imply that $b_{i,u} > b_{j,u}$. The next example lottery is illustrated in Table 6. The only difference in

Table 6. Second Lottery in the Example.

Probability	0.076	0.174	0.125	0.125	0.125	0.125	0.125	0.125
Outcome, $	10	3	-3	12	-4	-5	20	5

Table 7. Third Lottery in the Example.

Probability	0.125	0.05	0.05	0.125	0.039	0.361	0.125	0.125
Outcome, $	10	3	-3	12	-4	-5	20	5

these examples is the probability of outcomes. In this example, $\mathbb{E}\mathrm{U}_{10} = 0.176063$ and $\mathbb{E}\mathrm{U}_3 = 0.174$, however $b_{10,u} = 0.249927999$ and $b_{3,u} = 0.259190804$. Hence, expected utility approach favors information on outcome 10 whereas the ranking changes in buying price approach. This is another example why (a) of Proposition 9 does not hold in the opposite case. The final example that we analyze is illustrated in Table 7, which illustrates why (b) of Proposition 9 does not hold in the opposite case. When we rank information on outcomes 3 and -5, we observe that preference reversal occurs using difference in expected utility approach and buying price approach: $\mathbb{E}\mathrm{U}_3 = 0.05$ and $\mathbb{E}\mathrm{U}_{-5} = 0.06728$ whereas $b_{3,u} = 0.087598046$ and $b_{-5,u} = 0.070327026$. Information about a negative outcome is preferred in expected utility approach, however the decision maker pays more to obtain information on a positive outcome.

IV.2. Risk Aversion and the Value of Information

In this section, we establish how the decision maker's buying price for information about an outcome changes as he becomes more risk averse. We state the main result of this chapter below.

Proposition 10 *Let $u(x)$ and $v(x)$ be strictly concave and increasing utility functions such that (a) $r_u(x) \geq r_v(x)$ (b) Both $u(x)$ and $v(x)$ exhibit non-increasing absolute risk aversion. Then,*

(i) If $u_0 \geq \mathbb{E}\mathrm{U}$ and $v_0 \geq \mathbb{E}\mathrm{V}$, then $b_{i,v} \geq b_{i,u}$.

(ii) If $\pi_i > 0$, $u_0 \leq \mathbb{E}\mathrm{U}$ and $v_0 \leq \mathbb{E}\mathrm{V}$, then $b_{i,v} \leq b_{i,u}$.

Proof: Under (i), we consider two cases. First assume $\pi_i > 0$. The equations are

$$u(W) = \max(p_i \cdot u(W+\pi_i-b_{i,u})+(1-p_i)\cdot u(W-b_{i,u}), \sum_{k=1}^{n} p_k \cdot u(W+\pi_k-b_{i,u})) \quad (4.13)$$

$$v(W) = \max(p_i \cdot v(W+\pi_i-b_{i,v})+(1-p_i)\cdot v(W-b_{i,v}), \sum_{k=1}^{n} p_k \cdot v(W+\pi_k-b_{i,v})) \quad (4.14)$$

Since both utility functions exhibit non-increasing absolute risk aversion, $p_i \cdot u(W + \pi_i - b_{i,u}) + (1-p_i) \cdot u(W - b_{i,u}) > \sum_{k=1}^{n} p_k \cdot u(W + \pi_k - b_{i,u})$ and $p_i \cdot v(W + \pi_i - b_{i,v}) + (1-p_i) \cdot v(W - b_{i,v}) > \sum_{k=1}^{n} p_k \cdot v(W + \pi_k - b_{i,v})$. Hence, 4.13 and 4.14 reduce to

$$u(W) = p_i \cdot u(W + \pi_i - b_{i,u}) + (1 - p_i) \cdot u(W - b_{i,u}) \quad (4.15)$$

$$v(W) = p_i \cdot v(W + \pi_i - b_{i,v}) + (1 - p_i) \cdot v(W - b_{i,v}) \quad (4.16)$$

Let

$$F = \begin{cases} W + \pi_i - b_{i,u} & : \text{ w.p. } p_i \\ W - b_{i,u} & : \text{ w.p. } 1 - p_i \end{cases}$$

As the Equation 4.15 suggests $c(F, u) = W$ and since $r_u \geq r_v$, 4.16 implies $c(F, v) \geq W$, or in other words

$$p_i \cdot v(W+\pi_i-b_{i,u})+(1-p_i)\cdot v(W-b_{i,u}) \geq p_i \cdot v(W+\pi_i-b_{i,v})+(1-p_i)\cdot v(W-b_{i,v}) \quad (4.17)$$

Inequality 4.17 holds if and only if $b_{i,v} \geq b_{i,u}$.

Next, consider the case $\pi_i < 0$. The equations are

$$u(W) = \max(u(W-b_{i,u}), p_i \cdot u(W-b_{i,u}) + (1-p_i) \cdot \sum_{k \neq i} \frac{p_k}{1-p_i} \cdot u(W+\pi_k - b_{i,u})) \quad (4.18)$$

$$v(W) = \max(v(W-b_{i,v}), p_i \cdot v(W-b_{i,v}) + (1-p_i) \cdot \sum_{k \neq i} \frac{p_k}{1-p_i} \cdot v(W+\pi_k - b_{i,v})) \quad (4.19)$$

We cannot discern which term in the max operator should be dominating in 4.18 and 4.19, so we consider four subcases.

Subcase 1: $u(W) = u(W - b_{i,u})$ and $v(W) = v(W - b_{i,v})$. Clearly, this holds if and only if $b_{i,u} = b_{i,v} = 0$.

Subcase 2: $u(W) = u(W-b_{i,u})$ and $v(W) = p_i \cdot v(W-b_{i,v}) + \sum_{k \neq i} p_k \cdot v(W+\pi_k - b_{i,v})$. It is also clear that $b_{i,v} \geq b_{i,u} = 0$.

Subcase 3: $v(W) = v(W-b_{i,v})$ and $u(W) = p_i \cdot u(W-b_{i,u}) + \sum_{k \neq i} p_k \cdot u(W+\pi_k - b_{i,u})$. This case is the opposite of subcase 2, and we will show that this subcase leads to a contradiction. Note first $b_{i,v} = 0$. Hence, $v(W) \geq \sum_{k \neq i} \ (p_k/(1-p_i)) \cdot v(W+\pi_k)$. This implies $u(W - b_{i,u}) \geq \sum_{k \neq i} \ (p_k/(1-p_i)) \cdot u(W + \pi_k - b_{i,u})$ for $b_{i,u} \geq 0$ as $u(x)$ exhibits non-increasing absolute risk aversion. This is a contradiction. Hence, this subcase is not possible.

Subcase 4: $u(W) = p_i \cdot u(W - b_{i,u}) + \sum_{k \neq i} p_k \cdot u(W + \pi_k - b_{i,u})$ and $v(W) = p_i \cdot v(W - b_{i,v}) + \sum_{k \neq i} p_k \cdot v(W + \pi_k - b_{i,v})$. Consider $G = \{W + \pi_k - b_{i,u}, p_k\}_{k=1}^n$ with $\pi_i = 0$. Clearly, $c(G, u) = W$ and $c(G, v) \geq W$. This implies

$$p_i \cdot v(W - b_{i,u}) + \textstyle\sum_{k \neq i} p_k \cdot v(W + \pi_k - b_{i,u}) > \\ p_i \cdot v(W - b_{i,v}) + \textstyle\sum_{k \neq i} p_k \cdot v(W + \pi_k - b_{i,v})$$

which holds iff $b_{i,v} \geq b_{i,u}$. This finishes the proof of (i).

Under (ii), the equations are the following

$$\mathbb{E}\text{U} = \max(p_i \cdot u(W+\pi_i - b_{i,u}) + (1-p_i) \cdot u(W - b_{i,u}), \sum_{k=1}^{n} p_k \cdot u(W+\pi_k - b_{i,u})) \quad (4.20)$$

$$\mathbb{E}\mathrm{V} = \max(p_i \cdot v(W + \pi_i - b_{i,v}) + (1 - p_i) \cdot v(W - b_{i,v}), \sum_{k=1}^{n} p_k \cdot v(W + \pi_k - b_{i,v})) \quad (4.21)$$

We cannot reduce the Equations 4.20 and 4.21 further, so we need to consider four subcases

Subcase 1: $\mathbb{E}\mathrm{U} = p_i \cdot u(W + \pi_i - b_{i,u}) + (1 - p_i) \cdot u(W - b_{i,u})$ and $\mathbb{E}\mathrm{V} = \sum_{k=1}^{n} p_k \cdot v(W + \pi_k - b_{i,v}))$. Clearly, $b_{i,u} \geq b_{i,u} = 0$.

Subcase 2: $\mathbb{E}\mathrm{U} = \sum_{k=1}^{n} p_k \cdot u(W + \pi_k - b_{i,u}))$ and $\mathbb{E}\mathrm{V} = \sum_{k=1}^{n} p_k \cdot v(W + \pi_k - b_{i,v}))$. Equalities hold if and only if $b_{i,u} = b_{i,v} = 0$.

Subcase 3: $\mathbb{E}\mathrm{U} = p_i \cdot u(W + \pi_i - b_{i,u}) + (1 - p_i) \cdot u(W - b_{i,u})$ and $\mathbb{E}\mathrm{V} = p_i \cdot v(W + \pi_i - b_{i,v}) + (1 - p_i) \cdot v(W - b_{i,v})$. Let $L' = \{\pi_k, p_k/(1 - p_i)\}_{k=1,k \neq i}^{n}$. Then the original lottery can be equivalently represented for the decision maker with utility function $u(x)$ as follows.

$$L = \begin{cases} W + \pi_i & : \quad \text{w.p. } p_i \\ c(L', u) & : \quad \text{w.p. } 1 - p_i \end{cases}$$

Accordingly, the equation for $u(x)$ can be written as follows,

$$p_i \cdot u(W + \pi_i) + (1 - p_i) \cdot u(c(L', u)) = p_i \cdot u(W + \pi_i - b_{i,u}) + (1 - p_i) \cdot u(W - b_{i,u})$$

$$\frac{u(W + \pi_i) - u(W + \pi_i - b_{i,u})}{u(W - b_{i,u}) - u(c(L', u))} = \frac{1 - p_i}{p_i} \quad (4.22)$$

where $W + \pi_i \geq W + \pi_i - b_{i,u} \geq W - b_{i,u} \geq c(L', u)$. Since $r_v(x) \leq r_u(x)$, $\exists$ an increasing concave function ψ such that $u(x) = \psi(v(x))$ (see [24]). Then, by concavity and 4.22

$$\frac{v(W + \pi_i) - v(W + \pi_i - b_{i,u})}{v(W - b_{i,u}) - v(c(L', u))} \geq \frac{1 - p_i}{p_i},$$

or

$$\begin{aligned} (1 - p_i) \cdot v(W - b_{i,u}) + p_i \cdot v(W + \pi_i - b_{i,u}) &\leq p_i \cdot v(W + \pi_i) + (1 - p_i) \cdot v(c(L', u)) \\ &\leq p_i \cdot v(W + \pi_i) + (1 - p_i) \cdot v(c(L', v)) \\ &\leq \mathbb{E}\mathrm{V} \end{aligned}$$

$$\leq (1-p_i)\cdot v(W-b_{i,v})+p_i\cdot v(W+\pi_i-b_{i,v})$$

This holds if and only if $b_{i,u} \geq b_{i,v}$.

Subcase 4: $\mathbb{E}\mathrm{U} = \sum_{k=1}^{n} p_k \cdot u(W+\pi_k-b_{i,u}))$ and $\mathbb{E}\mathrm{V} = p_i \cdot v(W+\pi_i-b_{i,v})+(1-p_i)\cdot v(W-b_{i,v})$. We will show this case is not possible. Suppose that $p_i \cdot v(W+\pi_i-b_{i,v})+(1-p_i)\cdot v(W-b_{i,v}) > \sum_{k=1}^{n} p_k \cdot v(W+\pi_k-b_{i,v})$. This condition differentiates this subcase from subcase 2. Then, $v(W-b_{i,v}) > \sum_{k\neq i} (p_k/(1-p_i))\cdot v(W+\pi_k-b_{i,v})$. Since $b_{i,u}=0$, $\sum_{k\neq i} (p_k/(1-p_i))\cdot u(W+\pi_k) \geq u(W)$ which leads to $\sum_{k\neq i} (p_k/(1-p_i))\cdot v(W+\pi_k) \geq v(W)$. Since $v(x)$ exhibits non-increasing absolute risk aversion, this is a contradiction. Hence, under (ii), $b_{i,u} \geq b_{i,v}$. This completes the proof. □

If $\pi_i < 0$, $u_0 \leq \mathbb{E}\mathrm{U}$ and $v_0 \leq \mathbb{E}\mathrm{V}$, then it is not possible to come up with certain conclusions. Note that, by Proposition 9, the original equations reduce to

$$\begin{aligned} \mathbb{E}\mathrm{U} &= p_i \cdot u(W-b_{i,u}) + \sum_{k\neq i} p_k \cdot u(W+\pi_k-b_{i,u}) \\ \mathbb{E}\mathrm{V} &= p_i \cdot v(W-b_{i,v}) + \sum_{k\neq i} p_k \cdot v(W+\pi_k-b_{i,v}) \end{aligned} \tag{4.23}$$

Both sides of the equations evaluate a lottery. The right hand side of the equations in 4.23 evaluates the original lottery L and the right hand side of the first equation evaluates the lottery

$$H = \begin{cases} W-b_{i,u} & : \ \text{w.p. } p_i \\ H' & : \ \text{w.p. } 1-p_i \end{cases}$$

where $H' = \{W+\pi_k-b_{i,u}, p_k/(1-p_i)\}_{k=1,k\neq i}^{n}$. The equation for the decision maker with utility function $u(x)$ can be rewritten as

$$\begin{aligned} p_i \cdot u(W+\pi_i) + (1-p_i)\cdot u(c(L',u)) &= p_i \cdot u(W-b_{i,u}) + (1-p_i)\cdot u(c(H',u)) \\ \frac{p_i}{1-p_i} &= \frac{u(c(L',u)) - u(c(H',u))}{u(W-b_{i,u}) - u(W+\pi_i)} \end{aligned} \tag{4.24}$$

where $W + \pi_i \leq W - b_{i,u} \leq c(H', u) \leq c(L', u)$. From here 4.24, we obtain

$$\frac{p_i}{1 - p_i} \leq \frac{v(c(L', u)) - v(c(H', u))}{v(W - b_{i,u}) - v(W + \pi_i)} \tag{4.25}$$

If $v(c(L', v)) - v(c(H', v)) \geq v(c(L', u)) - v(c(H', u))$, then 4.25 implies

$$p_i \cdot v(W + \pi_i) + (1 - p_i) \cdot v(c(L', v)) \geq p_i \cdot v(W - b_{i,u}) + (1 - p_i) \cdot v(c(H', v))$$

$$p_i \cdot v(W - b_{i,v}) + \textstyle\sum_{k \neq i} \ (p_k/(1 - p_i)) \cdot v(W + \pi_k - b_{i,v}) \geq$$

$$p_i \cdot v(W - b_{i,u}) + (1 - p_i) \cdot \textstyle\sum_{k \neq i} \frac{p_k}{1 - p_i} \cdot v(W + \pi_k - b_{i,u})$$

which holds if and only if $b_{i,u} \geq b_{i,v}$. We have proved the following proposition.

Proposition 11 *Let $u(x)$ and $v(x)$ be strictly concave and increasing utility functions such that $r_u(x) \geq r_v(x)$. Consider the case with $\pi_i < 0$, $u_0 \leq \mathbb{E}\mathrm{U}$ and $v_0 \leq \mathbb{E}\mathrm{V}$. Then $b_{i,u} \geq b_{i,v}$ if $v(c(L', v)) - v(c(H', v)) \geq v(c(L', u)) - v(c(H', u))$, where $L' = \{\pi_k, p_k/(1 - p_i)\}_{k=1, k \neq i}^{n}$ and $H' = \{W + \pi_k - b_{i,u}, p_k/(1 - p_i)\}_{k=1, k \neq i}^{n}$.*

Following Propositions 10 and 11, we arrive at interesting conclusions. The less risk averse decision maker values information more when his original decision is to skip the lottery. In this case, the less risk averse decision maker seems to value an opportunity to change the decision more than the more risk averse decision maker. This makes sense as the certainty equivalent of the lottery is greater for the less risk averse decision maker, so despite the initial decision to skip the lottery, he is more willing to give a second chance to evaluate the lottery. The more risk averse decision maker does not value the lottery as much, and gives less value for a chance to reconsider the lottery. In opposite case, when the outcome in question is positive, it seems that the more risk averse decision maker is more nervous that the lottery could bring about unfavorable outcomes despite the initial favorable decision. Therefore, he values more to pursue further information to see if it is really necessary to play the lottery. The less risk averse decision maker seems to care less about such an information gathering activity.

IV.2.1. An Example on the Risk Aversion and the Value of Information

In this example, we compute the the buying price using the utility function $u(x) = -exp(-ax)$, where a is the coefficient of absolute risk aversion. First, we consider a lottery in which $u(W) \geq \mathbb{E}\text{U}$ and an outcome $\pi_i > 0$. In this case, the equation for $b_{i,u}$ is

$$U(W) = p_i \cdot u(W + \pi_i - b_{i,u}) + (1 - p_i) \cdot u(W - b_{i,u})).$$

Substituting the specific utility function and making necessary cancellations

$$1 = p_i \cdot \exp(-a \cdot \pi_i) \cdot \exp(a \cdot b_{i,u}) + (1 - p_i) \cdot \exp(a \cdot b_{i,u}) \tag{4.26}$$

Solving 4.26, we obtain

$$b_{i,u} = -\frac{1}{a} \cdot \ln(1 + p_i \cdot (\exp(-a \cdot \pi_i) - 1)) \tag{4.27}$$

Note that, $b_{i,u} > 0$ as $\pi_i > 0$, and the expression in 4.27 is independent of W as the decision maker's risk attitude is independent of W. In order to see the behavior of this expression with respect to a change in the degree of risk aversion, we compute its derivative

$$\frac{db_{i,u}}{da} = \frac{1}{a^2} \cdot [\frac{\pi \cdot p_i \cdot exp(-a \cdot \pi_i) \cdot a}{1 - p_i + p_i \cdot exp(-a \cdot \pi_i)} + \ln(1 + p_i \cdot (\exp(-a \cdot \pi_i) - 1))] \tag{4.28}$$

The first term in the brackets in Equation 4.28 is positive whereas the second term is negative. Proposition 10 states that 4.28 has to be negative, and we will verify this conclusion. After a little algebra, expression is negative if and only if

$$\frac{\pi_i \cdot p_i \cdot a}{exp(a \cdot \pi_i) \cdot (1 - p_i) + p_i} \leq a \cdot \pi_i + \ln(\frac{1}{exp(a \cdot \pi_i) \cdot (1 - p_i) + p_i}) \tag{4.29}$$

Two expressions in 4.29 are equal when $p_i = 0$ and $p_i = 1$. It is not possible to verify this inequality right away, so we check the derivatives with respect to p_i. Define

$$\partial LHS = d(\frac{\pi_i \cdot p_i \cdot a}{exp(a \cdot \pi_i) \cdot (1-p_i) + p_i})/dp_i$$

$$\partial RHS = d(a \cdot \pi_i + ln(\frac{1}{exp(a \cdot \pi_i) \cdot (1-p_i) + p_i}))/dp_i.$$

Then

$$\partial LHS = \frac{\pi_i \cdot a \cdot exp(-a \cdot \pi_i)}{[exp(a \cdot \pi_i) \cdot (1-p_i)]^2}$$

$$\partial RHS = \frac{exp(a \cdot \pi_i) - 1}{exp(a \cdot \pi_i) \cdot (1-p_i) + p_i}.$$

Note that, $\partial LHS|_{p_i} < \partial RHS|_{p_i}$ for $p_i \in [0, \bar{p_i})$ where

$$\bar{p_i} = \frac{exp(2a\pi_i) - exp(a\pi_i) - a\pi_i exp(a\pi_i)}{(exp(a\pi_i) - 1)^2}.$$

This inequality changes direction for $p_i \in [\bar{p_i}, 1]$. This and the equality of expressions in 4.29 implies that $db_{i,u}/da < 0$, which verifies the case (a) of Proposition 10 for $u(x) = -exp(-ax)$. Next, we illustrate that case (b) of Proposition 10 cannot be stated for $\pi < 0$ without the additional condition in Proposition 11. The equation for the buying price is

$$\mathbb{E}\text{U} = p_i \cdot u(W - b_{i,u}) + \textstyle\sum_{k \neq i} p_k \cdot u(W + \pi_k - b_{i,u})$$

Then, we substitute the utility function and simplify the expression

$$\sum_k p_k \cdot \exp(-a \cdot \pi_k) = \exp(a \cdot b_{i,u}) \cdot (p_i + \sum_{k \neq i} p_k \cdot \exp(-a \cdot \pi_k)) \tag{4.30}$$

If we solve 4.30 for $b_{i,u}$

$$b_{i,u} = \frac{1}{a} \cdot \ln(\frac{\sum_k p_k \cdot exp(-a \cdot \pi_k)}{p_i + \sum_{k \neq i} p_k \cdot exp(-a \cdot \pi_k)}) \tag{4.31}$$

Consider the lottery in table 8. When we set $a = 1$, $\mathbb{E}\text{U} > u_0$. If we compute the derivative of 4.31 with respect to a, $d(b_{-1,u})/da|_{a=1} < 0$ and $d(b_{-1.8,u})/da|_{a=1} > 0$. In

Table 8. Lottery for Illustration of Proposition 11.

Probability	0.125	0.125	0.05	0.2	0.125	0.125	0.125	0.125
Outcome, $	10	3	-1	12	-1.8	15	20	5

particular, $b_{-1,u}|_{a=1} = 0.10042363$ and $b_{-1,u}|_{a=1.01} = 0.099363105$ whereas $b_{-1.8,u}|_{a=1} = 1.210560037$ and $b_{-1.8,u}|_{a=1.01} = 0.210615481$. This shows that we need an additional condition to ensure that case (b) of Proposition 10 holds for $\pi_i < 0$, too.

CHAPTER V

VALUE OF INFORMATION IN INSURANCE RISK PROBLEMS

Consider a decision maker or enterprise ("the firm") at risk of losing an amount of capital (money) due to the occurence of unexpected events. Insurance coverage is available, at a premium, to partially compensate the firm should such a monetary loss occur. The firm must decide how much insurance to purchase. Clearly, a number of factors will influence the firm's decision, including the premium charged, the level of coverage available, the firm's utility function, and the risk of loss (e.g., probability distribution of loss). Before the insurance decision is made, the firm may, at a cost, gather data and facts that may help it refine its estimate of the likelihood of loss. Such data and facts are known colloquially as information, and we are interested in developing analytical methods to assess the value of information in reducing the expected total cost to the firm for insurance and for loss.

Insurance risk problems are a crucial aspect of enterprise risk management. Examples of insurance risk problems abound in every industry, and these issues are becoming increasingly important in production and supply chain management as production systems become "lean" and the firm relies more heavily on critical distribution and production capacity. Examples of unexpected events that disrupt production capability for which insurance is typically considered include plant fires, labor actions, earthquakes, supplier failures, and customs and transportation delays. As the enterprises concentrate on a more holistic framework for risk management, it is becoming increasingly important to consider multiple effects of each unexpected event. In this regard, it is crucial to determine the risk factors and their potential impacts on the enterprise.

We consider a single decision maker, the firm, whose objective is to determine the level of insurance purchased against a single factor that poses a risk of a loss. In this chapter, we address the effect of information on decision making in the context of optimizing the amount of coverage purchased, and thereby reducing expected cost. In this regard, we are interested in determining the most attactive information bundle among several alternatives, and in describing the set of information bundles that result in decreased expected costs. There is no uniform preference scheme among decision makers for different information bundles. Preference ranking of information bundles depends on the shape of utility function and the probability distribution. Different degrees of risk aversion may lead to different rankings of information bundles. However, as Ohlson [30] and Willinger [39] showed, it is possible to obtain a uniform ranking when the class of utility functions is appropriately reduced. In this chapter, we rank information bundles of different specific forms for a given utility function. Based on value of information results, we develop optimal insurance strategies for several different loss models that are relevant in production operations scenarios.

V.1. Notation and Problem Definition

In a risky environment, a firm must decide whether or not to purchase an insurance policy to protect its assets from loss, and if the decision is to purchase insurance, the firm must decide the level of coverage. The firm has a concave utility function $u : \mathbb{R} \to \mathbb{R}$ and an initial wealth W. An insurance package is available at a unit price q (≤ 1) which pays one unit per unit loss [1]. The firm may obtain certain information bundles prior to making the insurance decision.

Let $(\mathbb{R}, \mathcal{B}, \mu)$ be a probability space representing the risk that the firm has to manage, and let $X : \Omega \to \mathbb{R}$ be a random variable mapping the states of the world

[1] Note that we use common monetary units (e.g., \$, £, etc.) to represent unit loss, unit price, and unit insurance coverage.

into monetary outcomes. The random variable X denotes the level of monetary loss that the firm faces. Information is taken as a σ-algebra of events; that is, given an information bundle $\mathcal{G} \subset \mathcal{F}$, the firm is able to ascertain whether or not any of the events in $\mathcal{G}$ have occured. Based on available information, the firm updates its beliefs about the likelihood of loss. We assume that the information bundles obtained by the firm are not necessarily possessed by the insurance company; hence, the terms of the contract do not depend on the firm's information. However, this private information may induce the firm to change its decision about how much coverage to purchase, or even whether to purchase coverage or not.

In the subsequent sections, we will consider several loss scenarios to determine an optimal information and decision strategy.

V.2. Fixed Damage Level

We begin with a relatively simple scenario. Let us suppose initially that, if the loss occurs, it is a fixed amount d. Then the relevant probability space is given by Let $0 \leq \alpha \leq d$ be the units of coverage purchased, and denote the event that the damage occurs by $D \in \mathcal{F}$.

In the absence of information, the firm chooses the level of α that maximizes its expected utility. Let α^* be the optimum level of coverage. In this simple case,

$$X(\omega) = \begin{cases} W - \alpha q - d + \alpha & \text{when } \omega \in D, \\ W - \alpha q & \text{when } \omega \in D^c. \end{cases}$$

Then α^* can be expressed as

$$\alpha^* = \text{argmax}_\alpha (1 - \mu(D)) \cdot u(W - \alpha q) + \mu(D) \cdot u(W - \alpha q - d + \alpha).$$

If $\alpha^* > 0$, it satisfies the first-order condition

$$\mu(D)(1-q)\cdot u'(W-\alpha^* q - d + \alpha^*) = q(1-\mu(D))\cdot u'(W-\alpha^* q).$$

Now, we would like to determine the value of a particular information bundle vis-á-vis the level of insurance coverage purchased. Consider the simple information bundle $\mathcal{F}_A$ generated by $A \in \mathbb{R}$; i.e., $\mathcal{F}_A = \{A, A^c, \mathbb{R}, \emptyset\}$. Provided $E[|u(X)|] < \infty$, $E[u(X)|\mathcal{F}_A]$ is the unique function that satisfies (c.f. [10])

$$\begin{aligned}\int_A E[u(X)|\mathcal{F}_A]d\mu &= \int_A u(X)d\mu \\ &= \int_{A\cap D} u(X)d\mu + \int_{A\cap D^c} u(X)d\mu \\ &= u(W-\alpha q - d+\alpha)\cdot\mu(A\cap D) + u(W-\alpha q)\cdot\mu(A\cap D^c), \quad (5.1)\end{aligned}$$

and

$$\int_{A^c} E[u(X)|\mathcal{F}_A]d\mu = u(W-\alpha q-d+\alpha)\cdot\mu(A^c\cap D) + u(W-\alpha q)\cdot\mu(A^c\cap D^c). \quad (5.2)$$

From 5.1 and 5.2, it follows that

$$E[u(X)|\mathcal{F}_A] = \begin{cases} u(W-\alpha q-d+\alpha)\cdot\mu(D|A) + u(W-\alpha q)\cdot\mu(D^c|A) & \omega \in A \\ u(W-\alpha q-d+\alpha)\cdot\mu(D|A^c) + u(W-\alpha q)\cdot\mu(D^c|A^c) & \omega \in A^c \end{cases}$$

Accordingly, the firm has two different problems to solve. When A occurs, all the points in the sample space that belongs to event A^c becomes meaningles, so the firm updates the probability of the event D according to this information. After the probabilities are updated, the firm maximizes its expected utility

$$\max_\alpha u(W-\alpha q-d+\alpha)\cdot\mu(D|A) + u(W-\alpha q)\cdot\mu(D^c|A)$$

Similarly, when A^c occurs

$$\max_{\alpha} u(W - \alpha q - d + \alpha) \cdot \mu(D|A^c) + u(W - \alpha q) \cdot \mu(D^c|A^c)$$

The first order conditions yield

$$\frac{\mu(D|A)}{\mu(D^c|A)} = \frac{q}{1-q} \cdot \frac{u'(W - \alpha_A q)}{u'(W - \alpha_A q - d + \alpha_A)} \quad \omega \in A$$

and

$$\frac{\mu(D|A^c)}{\mu(D^c|A^c)} = \frac{q}{1-q} \cdot \frac{u'(W - \alpha_{A^c} q)}{u'(W - \alpha_{A^c} q - d + \alpha_{A^c})} \quad \omega \in A^c,$$

where α_A and α_{A^c} are optimums when A and A^c occurs, respectively. In no information case

$$\begin{aligned} E[u]|_{\alpha=\alpha^*} &= \mu(D) \cdot u(W - \alpha^* q - d + \alpha^*) + \mu(D^c) \cdot u(W - \alpha^* q) \\ &= [\mu(D|A) \cdot u(W - \alpha^* q - d + \alpha^*) + \mu(D^c|A) \cdot u(W - \alpha^* q)] \cdot \mu(A) \\ &\quad + [\mu(D|A^c) \cdot u(W - \alpha^* q - d + \alpha^*) + \mu(D^c|A^c) \cdot u(W - \alpha^* q)] \cdot \mu(A^c). \end{aligned}$$

Note that,

$$\max_{\alpha} E[u|\mathcal{F}_A](\omega) \geq \begin{cases} \mu(D|A) \cdot u(W - \alpha^* q - d + \alpha^*) + \mu(D^c|A) \cdot u(W - \alpha^* q) & \omega \in A \\ \mu(D|A^c) \cdot u(W - \alpha^* q - d + \alpha^*) + \mu(D^c|A^c) \cdot u(W - \alpha^* q) & \omega \in A^c \end{cases} \tag{5.3}$$

Two inequalities in 5.3 imply that $E[\max_\alpha E[u|\mathcal{F}_A]] \geq E[u]|_{\alpha=\alpha^*}$. Thus the firm is never worse off by obtaining more information for free. An information bundle is valuable to the firm when the firm reaches a different decision about the level of coverage purchased in the presence of the information. Otherwise, an information bundle is said to be *irrelevant*. In other words, if the firm makes the same decision with or without the information, the information is of no value to the firm.

The following proposition considers two special cases.

Proposition 12 *If $A \subset D$, then $\alpha_A = d$ and $\alpha_{A^c} \leq \alpha^*$, and if $A \supset D$, then $\alpha_A \geq \alpha^*$ and $\alpha_{A^c} = 0$.*

Proof: Suppose first that $A \subset D$. When A occurs, the firm's optimization problem is

$$\max_\alpha u(W - \alpha q - d + \alpha) \cdot \mu(D|A) + u(W - \alpha q) \cdot \mu(D^c|A) = \max_\alpha u(W - \alpha q - d + \alpha).$$

Since $q \leq 1$, $du/d\alpha \geq 0$. Hence, the more the units of coverage the firm has, the higher its utility, and it follows that $\alpha_A = d$. When A^c occurs, we have the usual conditions for optimum. Since $(\mu(D)/\mu(D^c)) \geq (\mu(D|A^c)/\mu(D^c|A^c))$, we have

$$\frac{u'(W - \alpha_{A^c} q)}{u'(W - \alpha_{A^c} q - d + \alpha_{A^c})} \leq \frac{u'(W - \alpha^* q)}{u'(W - \alpha^* q - d + \alpha^*)}.$$

Let

$$\frac{u'(W - \alpha q)}{u'(W - \alpha q - d + \alpha)} = g(\alpha).$$

Then

$$\frac{dg(\alpha)}{d\alpha} = \frac{-q \cdot u''(W - \alpha q) \cdot u'(W - \alpha q - d + \alpha) + (q - 1) \cdot u''(W - \alpha q - d + \alpha) \cdot u'(W - \alpha q)}{(u'(W - \alpha q - d + \alpha))^2}$$

Note that

$$-q \cdot u''(W - \alpha q) \cdot u'(W - \alpha q - d + \alpha) + (q - 1) \cdot u''(W - \alpha q - d + \alpha) \cdot u'(W - \alpha q) \geq 0,$$

which suggests that $dg(\alpha)/d\alpha \geq 0$. From here we may conclude that $\alpha^* \geq \alpha_{A^c}$, which demonstrates the first claim of the proposition.

Now assume that $A \supset D$. When A^c occurs, optimization problem becomes

$$\max_\alpha u(W - \alpha q - d + \alpha) \cdot \mu(D|A^c) + u(W - \alpha q) \cdot \mu(D^c|A^c) = \max_\alpha u(W - \alpha q).$$

Then, clearly $\alpha_{A^c} = 0$. When A occurs, we have

$$\frac{\mu(D)}{\mu(D^c)} \leq \frac{\mu(D|A)}{\mu(D^c|A)}.$$

Using the FOC for optimum, we then conclude that $\alpha_A \geq \alpha^*$. $\square$

Intuitively, the proposition demonstrates that when A is a subset of D, the firm will be able to conclude that when A occurs damage will surely obtain, and since $q \leq 1$, the optimal decision for the firm is to purchase full coverage. On the other hand, when A^c occurs, the likelihood of loss is smaller, and hence, the optimal decision of the firm is to reduce the level of coverage from the original case with no information. However, when D is a subset of event A, the optimal level of coverage is expected to move in the opposite direction. When A occurs, the likelihood of the shock increases, so the firm should purchase more coverage. Instead, when A^c occurs, the firm learns with certainty that the shock will not obtain, so no coverage is purchased at all.

V.3. Random Damage Level

A more realistic case is where the level of damage is random. For mathematical ease, we would like to consider the case where $d : \Omega \to \mathbb{R}$ is a continuous uniform random variable with range $[0, W/n]$. Other distributions can be assumed too, however, the computations can easily become complicated and this complication does not add much to the demonstration of the value of information. Here, n is a parameter determining how influential the maximum damage is on the wealth level of the decision maker. Assume that the decision maker's utility function is $u(x) = x - \frac{1}{2W}x^2$, in which x is the level of wealth. It is easy to check that this function satisfies the properties of a utility function. The objective is to decide the fraction of the damage that will be restored back to the decisiom maker by the insurance company in case a damage obtains. Hence, different from the earlier case where damage was fixed, α is the fraction of damage covered. We assume that the cost of coverage is, $c(\alpha) = \alpha W/b$. The higher the value of b, the cheaper the coverage is. We assume that $b \geq n$, so that the full coverage never costs strictly greater than any possible level of damage.

Again, we would like to rank simple information bundles of the form $\mathcal{F}_{Aa} = \{A_a, A^c{}_a, \mathbb{R}, \emptyset\}$, generated by the event $A_a = \{d \leq a\}$. Let $v_\alpha(W, d, b) = u(x)|_{x=W-\alpha q-d+\alpha}$. Then

$$v_\alpha = \frac{W}{2}(1 - \frac{\alpha^2}{b^2}) - \frac{\alpha}{b}(1-\alpha)d - \frac{1}{2W}(1-\alpha)^2 d^2.$$

First, we would like to determine the level of coverage purchased when information $\mathcal{F}_{Aa}$ is obtained. When A_a occurs (i.e. when $\omega \in A_a$)

$$E[v_\alpha|\mathcal{F}_{Aa}](\omega) = \frac{W}{2}(1 - \frac{\alpha^2}{b^2}) - \frac{\alpha}{b}\frac{(1-\alpha)a}{2} - \frac{(1-\alpha)^2}{6W}a^2.$$

We maximize this with respect to α. After taking the first order conditions, the unconstrained optimal level of α is calculated as

$$\alpha_{u,A_a} = (\frac{a^2}{3W} - \frac{a}{2b})/(\frac{W}{b^2} - \frac{a}{b} + \frac{a^2}{3W}). \tag{5.4}$$

When $A^c{}_a$ occurs

$$E[v_\alpha|\mathcal{F}_{Aa}](\omega) = \frac{W}{2}(1 - \frac{\alpha^2}{b^2}) - \frac{\alpha}{b}\frac{(1-\alpha)}{2}(\frac{W}{n} + a) - \frac{(1-\alpha)^2}{6W}(\frac{W^2}{n^2} + a\frac{W}{n} + a^2)$$

Similarly, the unconstrained optimal α is calculated as

$$\alpha_{u,A^c{}_a} = (-\frac{1}{2}(\frac{W}{n} + a) + \frac{\frac{W^2}{n^2} + a\frac{W}{n} + a^2}{3\frac{W}{b}})/(\frac{W}{b} - (\frac{W}{n} + a) + \frac{\frac{W^2}{n^2} + a\frac{W}{n} + a^2}{3\frac{W}{b}}). \tag{5.5}$$

Since α_{A_a}, $\alpha_{A^c{}_a} \in [0, 1]$, we need to analyze how α_{u,A_a} and $\alpha_{u,A^c{}_a}$ behave for different values of a. The behavior depends on whether a is greater than W/b or not, as the following theorem suggests.

Theorem 7 *Suppose that the decision maker with a utility function $u(x)$ learns the damage level in advance. Then he purchases no coverage when $d < W/b$ and full* *coverage when $d \geq W/b$.*

Proof: The decision maker's utility is $u(W - c(\alpha) - d + \alpha d) = u(W(1 - (\alpha/b)) - d(1-\alpha))$. Since $du/d\alpha = u'(W(1-(\alpha/b)) - d(1-\alpha)) \cdot (d - (W/b))$, the result follows after observing that $u'(W(1-(\alpha/b)) - d(1-\alpha)) \geq 0$ $\square$

In light this theorem, we will assume first that $a < W/b$. From Theorem 7, we know that α_{A_a}=0. This can be verified from 5.4. The denominator is positive and the numerator is negative.

$$\frac{W}{b^2} - \frac{a}{b} + \frac{a^2}{3W} = \frac{1}{b}(\frac{W}{b} - a) + \frac{a^2}{3W} \geq 0 \text{ and } a(\frac{a}{3W} - \frac{1}{2b}) \leq 0$$

implying that $\alpha_{u,A_a} \leq 0$. Since $E[v_\alpha(X)|\mathcal{F}_{Aa}](\omega)$ is a quadratic function of α for $\omega \in A$ and $d^2(E[v_\alpha|\mathcal{F}_{Aa}](\omega))/d\alpha^2 = -$ denominator ≤ 0, α_{u,A_a} is the unique maximizer. Consequently the constrained maximum is attained at $\alpha_{A_a} = 0$. The following proposition gives the value of $\alpha_{A^c{}_a}$ under different circumstances.

Proposition 13 *Assume $a < W/b$. When $A^c{}_a$ occurs, the optimal level of coverage is as follows,*

For $n \leq b \leq \frac{3}{2}n$:

$$\alpha_{A^c{}_a} = \begin{cases} 0 & : \quad 0 \leq a \leq \tilde{a} \\ \alpha_{u,A^c{}_a} & : \quad \tilde{a} \leq a \leq \bar{a} \\ 1 & : \quad \bar{a} \leq a < \frac{W}{b} \end{cases}$$

For $\frac{3}{2}n \leq b \leq 2n$:

$$\alpha_{A^c{}_a} = \begin{cases} \alpha_{u,A^c{}_a} & : \quad 0 \leq a \leq \bar{a} \\ 1 & : \quad \bar{a} \leq a < \frac{W}{b} \end{cases}$$

And, for $b \geq 2n$: $\qquad \alpha_{A^c{}_a} = 1$

where $\bar{a}$ is a solution to

$$\frac{W}{b} = \frac{1}{2}(\frac{W}{n} + a)$$

and

$$\tilde{a} = \frac{3}{4}\frac{W}{b} - \frac{1}{2}\frac{W}{n} + (\sqrt{\frac{b}{3n}(1-\frac{b}{n}) + \frac{1}{4}})\frac{3}{2}\frac{W}{b}.$$

Proof: Denote the denominator of 5.5 by d_a and the numerator by n_a. We will first show that d_a is always positive. The value of a that minimizes d_a, keeping other parameters fixed is given calculated by solving $dd_a/da = 0$ for a, which yields

$$a_{min} = \frac{W}{b}(\frac{3}{2} - \frac{b}{2n}).$$

Note that, $d^2 d_a/da^2 \geq 0$, so a_{min} is a minimizer. However, for $b \geq 3n$, $a_{min} \leq 0$, so d_a is actually minimized at $a = 0$. Hence, for $b \geq 3n$,

$$d_a \geq d_a|_{a=0} = \frac{W}{b} + \frac{W}{n}(\frac{b}{3n} - 1) \geq 0.$$

For $n \leq b \leq 3n$, d_a is minimized at $a = a_{min}$. Hence, the minimum value for the denominator is

$$d_a|_{a=a_{min}} = \frac{1}{4}\frac{W}{b} + \frac{1}{4}\frac{Wb}{n^2} - \frac{1}{2}\frac{W}{n}.$$

We need to see if this is positive for $n \leq b \leq 3n$. Since

$$\frac{dd_a|_{a=a_{min}}}{db} = \frac{1}{4}W(\frac{1}{n^2} - \frac{1}{b^2}) \geq 0,$$

it suffices to see that $d_a|_{a=a_{min}} \geq 0$ at $b = n$. We have $d_a|_{a=a_{min},b=n} = 0$, so $d_a \geq 0$ for $n \leq b \leq 3n$. Hence, the denominator is always positive. This implies that $\alpha_{u,A^c{}_a}$ is always a unique unconstrained maximizer.

We need to analyze the numerator. Observe that for $b \geq 2n$, $n_a \geq d_a$, hence the constrained maximum, $\alpha_{A^c{}_a} = 0$. Now, assume that $\frac{3}{2}n \leq b \leq 2n$. Then $n_a \geq d_a$ whenever $a \geq \bar{a}$ where $\bar{a}$ is a solution to $W/b = 1/2\ (W/n + a)$. Furthermore,

$$\frac{dn_a}{db} = \frac{1}{3}(\frac{W}{n^2} + \frac{a}{n} + \frac{a^2}{W}) \geq 0 \quad \text{and} \quad n_a|_{b=\frac{3}{2}n} = \frac{1}{2}\frac{a^2 n}{W} \geq 0,$$

so $n_a \geq 0$. From here it follows that $\alpha_{A^c{}_a} = \alpha_{u,A^c{}_a}$ for $a \leq \bar{a}$. Finally, assume that

$n \leq b \leq \frac{3}{2}n$. It is still true that $n_a \geq d_a$ for $a \geq \bar{a}$. However, n_a becomes negative for values of $0 \leq a \leq \tilde{a}$ where

$$\tilde{a} = \frac{3}{4}\frac{W}{b} - \frac{1}{2}\frac{W}{n} + (\sqrt{\frac{b}{3n}(1 - \frac{b}{n}) + \frac{1}{4}})\frac{3}{2}\frac{W}{b}.$$

From here, the result follows. □

The other case we need to consider is $a \geq W/b$. We know from Theorem 7 that the optimal level of α is 1 when A_a^c occurs. As earlier, one can verify this result from the expression of the unconstrained optimum. The following proposition tells us about the optimal level of coverage when A_a occurs

Proposition 14 *Assume $a \geq W/b$. When A_a occurs, the optimal level of coverage is as follows,*

$$\alpha_{A_a} = \begin{cases} 0 & : \frac{W}{b} \leq a \leq \frac{3W}{2b} \\ \alpha_{u,A_a} & : \frac{3W}{2b} \leq a \leq \frac{2W}{b} \\ 1 & : \frac{2W}{b} \leq a \leq \frac{W}{n} \end{cases}$$

Proof: Let n_a and d_a be the numerator and the denominator of the 5.4 respectively. First of all d_a is minimized at $a = 3W/2b$ and $d_a|_{a=\frac{3W}{2b}} = 0$. This implies $d_a \geq 0$. Furthermore, $n_a \leq 0$ for $a \leq 3W/2b$. Thus, $\alpha_{A_a} = 0$ for $a \leq 3W/2b$. One can also verify very easily that $n_a \geq d_a$ for $a \geq 2W/b$. Hence, the result follows. □

One remark that we should immediately make about the proposition above is that it is never optimal to get full coverage if $W/n < 2W/b$. This case occurs when the coverage is very costly, and the maximum level of damage is low.

V.4. Information with Increased Precision

Another problem related with the previous example is to compare information bundles of some increased precision. It is possible purchase information bundles of the form $\mathcal{F}_r = \sigma(A_{1,r}, ..., A_{i,r}, ..., A_{r,r})$ where $A_{i,r} = [W/(rn(i-1)), W/rni]$. The sample space for level of damage is partitioned into intervals of equal length, and the decision maker learns which interval the actual level of damage belongs in advance when he buys the information bundle. After acquiring the information, the optimal level of coverage can be determined easily in most cases from Theorem 7. The only interesting case is when decision maker learns that A_i occurs where

$$(i-1)\frac{W}{n} \leq \frac{W}{b} \leq i\frac{W}{n}.$$

The form of the utility function and the distribution of the damage remains the same in this new setting. Then, when A_i occurs, the expected utility for the decision maker is calculated as

$$E[v_\alpha|\mathcal{F}_r](\omega) = \frac{W}{2}(1-\frac{\alpha^2}{b^2}) - (1-\alpha)\frac{\alpha}{b}\frac{W}{2rn}(2i-1) - \frac{(1-\alpha)^2 W}{6r^2n^2}(3i^2-3i+1)$$

Again, from first order conditions, the unconstrained optimum is

$$\alpha_{u,r} = (-\frac{W}{2brn}(2i-1) + \frac{1}{3}\frac{W}{r^2n^2}(3i^2-3i+1))/(\frac{W}{b^2} - \frac{W}{brn}(2i-1) + \frac{1}{3}\frac{W}{r^2n^2}(3i^2-3i+1)) \tag{5.6}$$

We know from Theorem 7 that the constrained optimum is easy to obtain when A_i occurs if

$$(i-1)\frac{W}{n} \leq \frac{W}{b} \leq i\frac{W}{n}$$

does not hold.

Corollary 7 *(i) If $W/b \geq iW/rn$, then $\alpha_{i,r} = 0$ (ii) If $W/b \leq (i-1)W/rn$, then $\alpha_{i,r} = 1$.*

Proof: In the first case, the decision maker learns that the damage level is less than W/b and in the second case, he learns instead that the damage level is greater than W/b. The result then follows from Theorem 7. $\square$

In the next proposition, we determine the optimal level of coverage when

$$(i-1)\frac{W}{rn} \leq \frac{W}{b} \leq i\frac{W}{rn}$$

holds. Let k be a real number satisfying $b = krn$. Then, the level of coverage when the level of damage belongs to the interval containing W/b is as follows.

Proposition 15 *Let*

$$(i-1)\frac{W}{n} \leq \frac{W}{b} \leq i\frac{W}{n}.$$

Then the optimal level of coverage when A_i occurs is as follows,

$$\alpha_{i,r} = \begin{cases} 0 & : \quad k \in [\frac{1}{i}, \bar{k}] \\ \alpha_{u,r} & : \quad k \in [\bar{k}, \hat{k}] \\ 1 & : \quad k \in [\hat{k}, \frac{1}{i-1}] \end{cases}$$

where

$$\bar{k} = \frac{6i-3}{6i^2-6i+2} \text{ and } \hat{k} = \frac{1}{i-\frac{1}{2}}.$$

Proof: Let n_r and d_r be the numerator and the denominator of 5.6 respectively. First, we would like to see whether the denominator is always positive or not. Assume first that $k \in [1, \infty)$. This means $W/b \in [0, W/rn]$ and A_1 occurs. Then

$$d_r|_{i=1} = \frac{W}{3r^2n^2} + \frac{W}{r^2n^2}\frac{1}{k}(1-\frac{1}{k}) \geq 0.$$

Next assume that $W/b \in [(i-1)W/rn, iW/rn]$ for some $1 < i \leq r$. Then, after taking the first order conditions, one can see that $b_{min} = 2rn/(2i-1)$ is the minimizer of the denominator and

$$(i-1)\frac{W}{rn} \leq \frac{W}{b_{min}} \leq i\frac{W}{rn}.$$

Finally, $d_r|_{b=b_{min}} = W^2/12r^2n^2 \geq 0$, so d_r is always positive.

Next,

$$n_r|_{b=krn} = -\frac{Wi}{kr^2n^2} + \frac{W}{2kr^2n^2} + \frac{Wi^2}{r^2n^2} - \frac{Wi}{r^2n^2} + \frac{W}{3r^2n^2}.$$

Furthermore,

$$\frac{dn_r|_{b=krn}}{dk} = \frac{W}{k^2r^2n^2}(i - \frac{1}{2}) \geq 0,$$

so the numerator is monotonically increasing in k. Then solving the inequality $n_r|_{b=krn} \leq 0$, we obtain that $n_r|_{b=krn}$ is negative for $k \leq \bar{k}$. Furthermore $n_r \geq d_r$ for $k \geq \hat{k}$. The result follows from these observations. $\square$

V.5. Numerical Results

In this section, we will discuss the effects of changes in the level of parameters in the both models related to random damage level. First, we will present the results of simulations on the model where we only consider information bundles formed by the events of the form $A_a = \{d \leq a\}$. Three parameters are crucial here: a, b and n. a determines how the sample space is partitioned after acquisition of the information bundle. From Theorem 7, we know that the main concern of he decision maker is to learn whether the actual damage level, d, is greater than W/b or not. Thus, the relation between a and b is particularly important for the value of information.

In what follows, we fix the level of wealth at 200 and fixed the value of n at 4. Keeping the values of W and n, we get a picture of how the value of information behaves as a function of b as a assumes different values. The numerical results support the conclusion that the maximum value of information is attained at $a = W/b$, for each value of b. Figures 5 and 6 illustrate the value of information for different values of b. As the level of a approaches from 0 to W/b, the value of information amplifies whereas as the level of a drifts away from W/b, the value of information diminishes. It is possible to observe that on both ends of the spectrum, as a drifts further away from

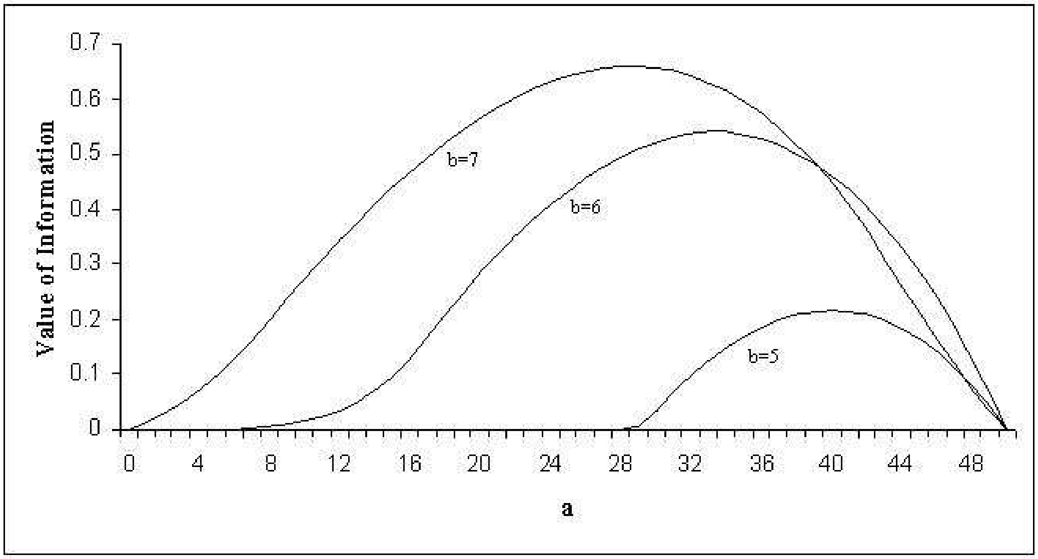

Fig. 5. Value of Information as a Function of "a" for Different Fixed Values of "b".

W/b, information becomes valueless. This is mainly due to noise factors that do not provide a better description of the environment compared to the no information case. As a result, the decision maker has no reason to manipulate the initial decision on the level of coverage purchased.

Figures 7 and 8 illustrate the value of information for different values of a. The shape of the graphs in each figure remains the same; however at different fixed values of a, the overal magnitude of the value of information differs. There is one intuitive explanation to this change in the overall magnitude. The increase in the value of b is good for the decision maker because the cost of coverage declines as b increases. Hence, the utility level of the decision maker is positively affected from an increase in the level of b. As a result of the decline in the cost, the decision maker purchases more coverage. The cost of making a wrong decision declines, hence the information becomes less valuable to the decision maker. The second effect is stronger for greater values of b. The behavior of the value of information as a function of a for different fixed values of b can be attributed to this. Based on the results, the most valuable

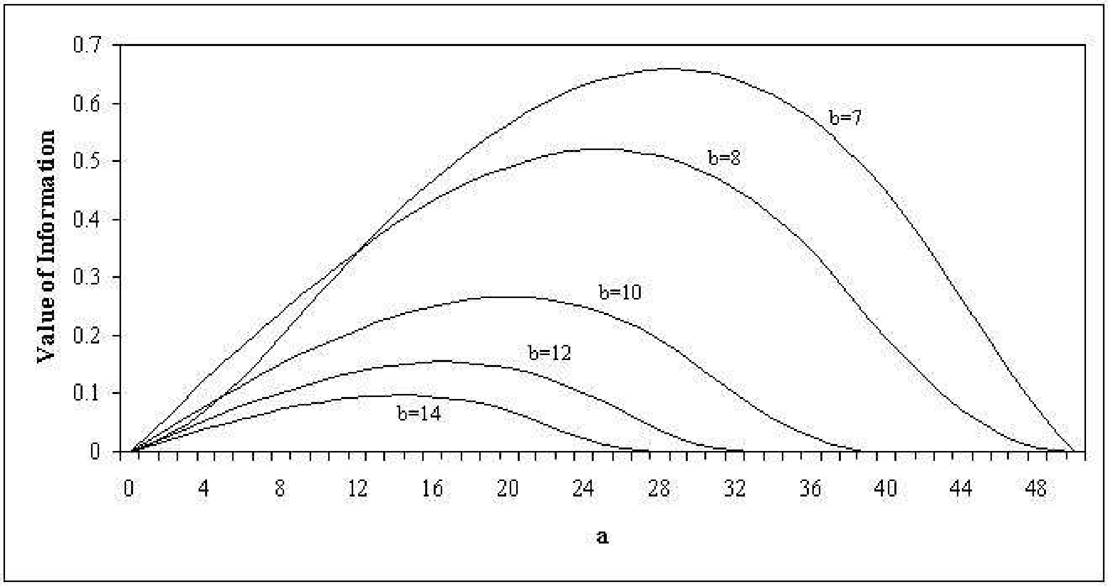

Fig. 6. Value of Information as a Function of "a" for Different Fixed Values of "b".

information bundle among this class is $\mathcal{F}_{\frac{W}{b}}$.

The evolution of the behavior of the value of information in Figures 5, 6, 7 and 8 has a similar picture for different values of W and n. The value of information is a always maximum at $a = W/b$. Accordingly, the changes in W and n only changes the value of a that the value of information attains the maximum value.

The support of the distribution of the level of damage is controlled by the parameter n. As n increases, the support shrinks, high levels of damage fail to be a possibility. Not surprisingly, this induces the decision maker to reduce his purchase of the coverage. At high levels of n, no coverage is purchased at all, even when information is obtained. Thus, the value of information tends to zero. In Figure 9, we have taken the values for W and b as 2000 and 20 respectively. One observation is that the value of information increases until a particular value of n is reached. Furthermore, this increase is linear. For smaller values of n, the decision maker purchases full coverage when he obtains no information. When some information bundle is acquired, if $a \leq W/b$, then his decision remains the same for small values of n, i.e. purchase no coverage when A occurs and

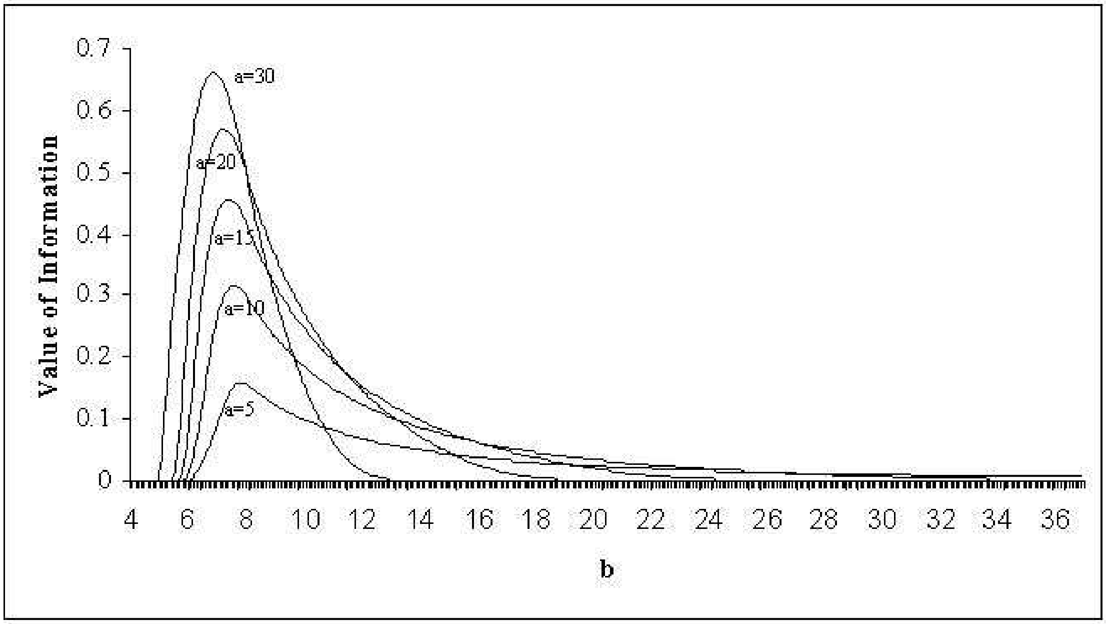

Fig. 7. Value of Information as a Function of “b” for Different Fixed Values of “a”.

full coverage when A^c occurs. As a result, the value of information changes linearly as the likelihoods of A and A^c change linearly with n, not because the decision maker comes up with a new decision when these events occur. After some threshold value, the decision maker begins to reduce his purchase of coverage in no information case and this linear relation is lost. For high values of n, the decision maker reduces the level of coverage purchased even further, and the value of information hits zero when the value of n is high enough to make sure that he purchases no coverage even when A^c occurs. When $a \geq W/b$, similar behavior is observed. In that case, for high values of n, the the decision maker takes advantage of further information when A^c occurs. The possibility of this advantage diminishes as n increases, so does the value of information.

Next, we would like to analyze the second model numerically. The parameter r controls how precise the information is. Greater values of r refers to higher precision. The natural question is to see if it is possible to find a monotonic relation between the value of information and r. Such a relation fails to exist, because the information bundle with a higher precision does not necessarily contain the information in the

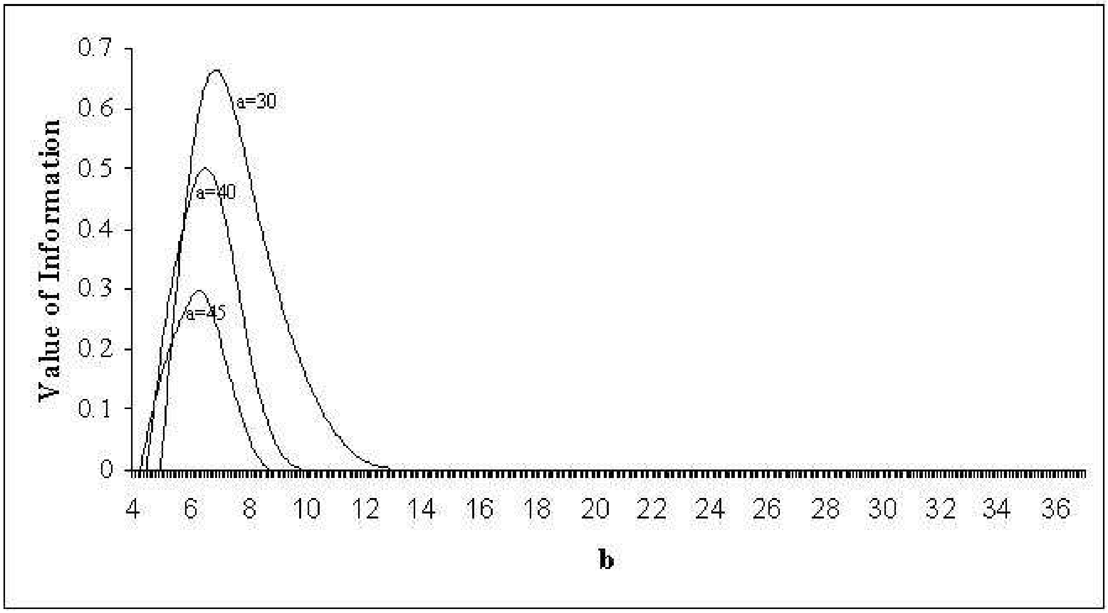

Fig. 8. Value of Information as a Function of “b” for Different Fixed Values of “a”.

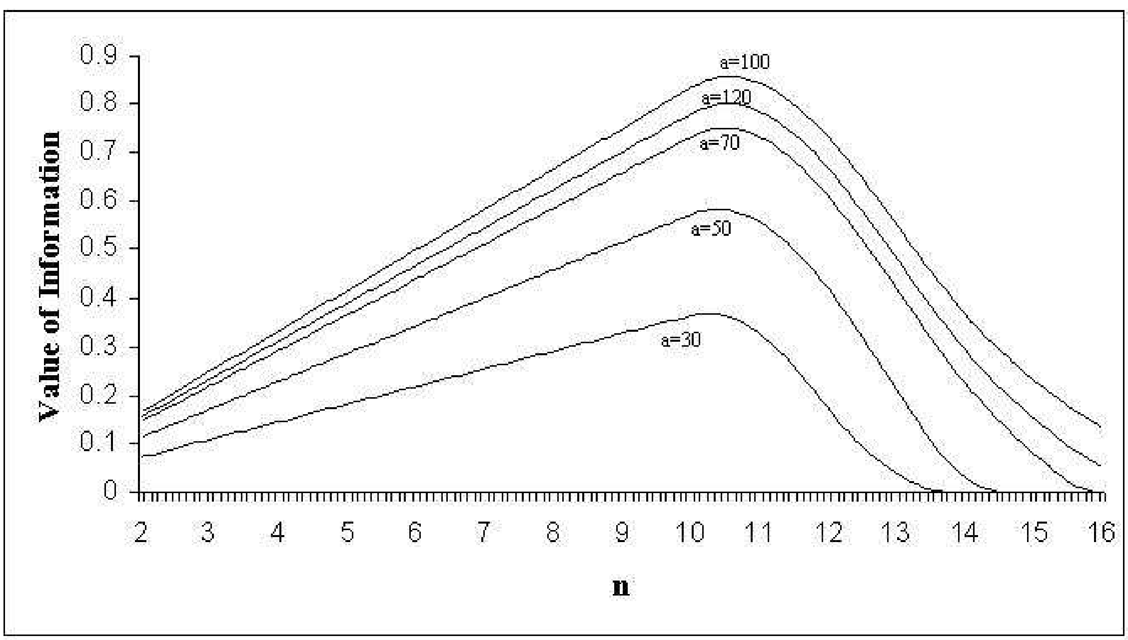

Fig. 9. Value of Information as a Function of “n” for Different Fixed Values of “a”.

Table 9. Value of Information with Doubled Precision.

W	n	b	r	Value of information
100	4	6	2	0.217014
100	4	6	4	0.254991
100	4	6	8	0.26652
100	4	6	16	0.269148
100	4	6	32	0.269837
100	4	6	64	0.270005

bundle with a lower precision. Both bundles may provide different information, none of which strictly contains the other. The information bundle with a higher precision might exclude some crucial information that exist in a bundle with a lower precision. Table 9 illustrates two cases where a monotonic relation is observed, simply because the information bundle with a higher precision includes the bundle with a lower precision. On the other hand, Table 10 illustrates that increased precision does not necessarily imply higher value of information.

The behavior of the value of information as a function of b depends on the fixed values of r, n and W. However, to observe the general behavior, it suffices to fix W and r and observe the behavior for different values of n, because the relation mainly depends on where W/b fits in the partition of $[0, W/n]$ by the information bundle with precision r. In Figure 10, $W = 2000$ and $r = 8$. When W/b fits in the very first interval in the partition, information may not be valuable. In all the graphs in Figure 10, the value of information converges to zero as W/b gets small and the decision taken by the decision maker is the same as in no information case even when it is learned that the level of damage is not going to be high. Another interesting remark is that

Table 10. Value of Information with Increasing Precision.

W	n	b	r	Value of information
100	2	7	2	0.0294349
100	2	7	3	0.185815
100	2	7	4	0.189998
100	2	7	5	0.170748
100	2	7	6	0.185815
100	2	7	7	0.194363
100	2	7	8	0.189998

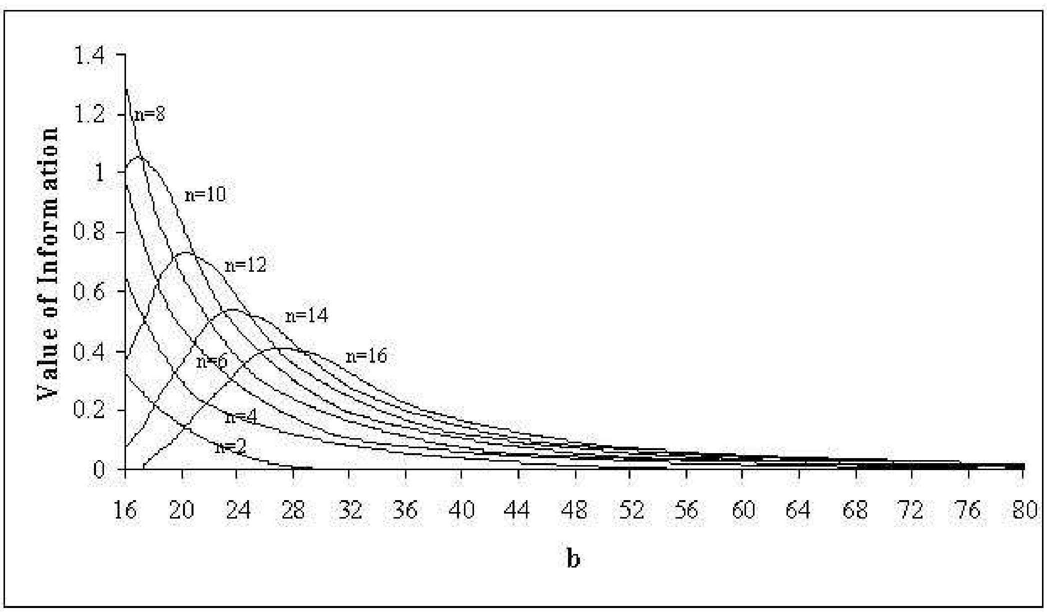

Fig. 10. Value of Information as a Function of “b” for Different Fixed Values of “n”.

as n larger, the value of information tends to increase as b increases until it reaches a peak point and then starts to diminish, whereas for smaller values of n, the value of information diminishes without revealing any increasing trend initially. The graphs for smaller values of n are a little bit misleading because in Figure 10 we don't allow b to take values less than 16. This is because we compare different graphs for different levels of n and in our analysis we never allow $b > n$ for in that case analysis becomes trivial. In fact, the value of information has an increasing trend for smaller values of b and after reaching the peak point, it gradually diminishes. As W/b moves to the middle of $[0, W/b]$, the decision maker purchases partial coverage. Then the information becomes more valuable for the decision maker updates his initial decision when any of the events in the information bundle occurs. Therefore, we observe an increase in the value of information as b increases for smaller values of b.

CHAPTER VI

VALUE OF INFORMATION FOR CORPORATE INSURANCE DECISIONS

In the current era of globalization, multinational firms continue to expand into emerging global markets. In order to bring better service to different markets, each of which offers high potential profits, firms build new facilities in economically developing countries. Accordingly, the enterprises encounter many operational and financial decisions in an unfamiliar environment. Information plays a crucial role in reducing uncertainty in this unfamiliar environment, and can help the enterprise to eliminate some unnecessary risks and to identify the likelihood and severity of risks retained. The calculation of fair premium and determination of how much insurance is required against a specified risk is possible if the decision makers can form a clear picture of risks to which the enterprise is exposed.

In this chapter, we focus on determining the value of information in making insurance decisions about physical hazard risks. The risks involve any physical events that would limit the capacity of the facility for a specified period time, e.g. fire, flood, utility outage, earthquake etc. These events may not impose a full restriction on the production capacity of the facility. Rather, they may remove a percentage of the available capacity based on the severity of the risk event. We take the capacity as fixed for the whole facility, ignoring the production and inventory decisions within the facility that actually determine how the capacity is utilized. The total capacity of the facility is allocated to a single product that serves a single market.

We consider the problem of assessing the value of information in insuring physical risks for a single facility that serves a single market and is exposed to multiple risk factors. There is a certain cost per unit of item delivered. The price of each item depends on the market conditions and the costs of item delivery. The criticality of operations in the facility depends on the demand patterns for each product in the markets served, the speed of recovery from the impacts of each risk factor, the revenue goals and the strategic goals to expand the market share. While physical property damage may occur at a facility due to a risk event, the business interruption due to a reduction in capacity can have far more financially severe consequences. The enterprise should incorporate the business interruption risk in their property and casualty insurance decisions. Therefore, we use lost sales and lost production capability as a proxy to measure the potential loss due to each risk factor.

VI.1. The Model

We consider a risk process consisting of two independent types of risk events each of which reduces the capacity at the facility to a fraction of the full capacity. When the facility operates at its full capacity, it can produce C units of a single product per unit time. Therefore, the facility is able to produce Ct units in t time units. We assume that the production serves a single market and the sale price of the product is p. The cost of producing a single item is c. This cost includes all the production and and distributional costs associated with a single item. Therefore, the firm is exposed to an economic loss of $(p - c)$ dollars per unit of unsatisfied demand. The main risk that each risk event imposes on the facility is the lost demand due to a reduction in the capacity. In this vein, we ignore the physical damage on the facility in insurance decisions.

Let $(\Omega, \mathcal{F}, P)$ be a probability space which represents the set of states of the world and the probability law that governs this abstract space. All the random phenomenon in this model is defined on this probability triple. We will assume $\mathcal{F}$ includes all the singleton subsets of Ω. The uncertainty in the model is due to stochastic severity level from a risk event and the time it takes for the next risk event to hit the facility. When the facility is hit by a risk event of type j, it enters a fixed recovery period of S_j. It is assumed that a risk event of the same type cannot hit the facility during the recovery period.

Definition 10 *The risk process is the stochastic tuple* $\{X_n^j, R_n^j\}_{j=1,2;n\in\{-\infty,\dots,-1,0,1,..,m_j\}}$, *where* $X_n^j \in (-\infty, 1]$, $R_n^j \in [0,1]$, *and*

$m_j = max\ \{k : k \cdot S_j \leq 1\}$

$X_n^j, n > 0 :$ *Time of the* n^{th} *risk event of type* j *after time* $t = 0$.

$X_n^j, n < 0 :$ *Time of the* $(-n+1)^{st}$ *risk event of type* j *before time* $t = 0$.

$R_n^j : \Omega \rightarrow [0,1] :$ *The fractional remaining capacity left when* n^{th} *risk event of type* j *hits the facility.*

The decision to purchase insurance is taken at time $t = 0$. Hence, we count the risk events from the origin, $t = 0$. The tuples that are negatively enumerated form the history of the process whereas the positively enumerated tuples forms the risks that the firm transfers to the insurance company during the contract horizon. The time between each risk event includes the fixed recovery time from the earlier risk event and the time to next risk event after the facility recovers.

During the recovery period, the production rate is reduced by $100 \cdot (1 - R_n^j)\%$. We define $\{P(t), t \geq 0\}$ to be the production rate process where $P(t) \in [0, C]$. Each sample path of $\{P(t), t \geq 0\}$ is a step function with finite number of jumps in any bounded set at risk event epochs and recovery epochs. We assume that the facility sets $P(t) = C$ when there is no recovery operation from a risk event. When a risk event with fractional capacity level R_n^j arrives at time t, the capacity is reduced to $R_n^j \cdot P(t^-)$. It is possible that a risk event of type j arrives while the facility is recovering from a risk event of type i. In this case, capacity is reduced by the multiple of fractional

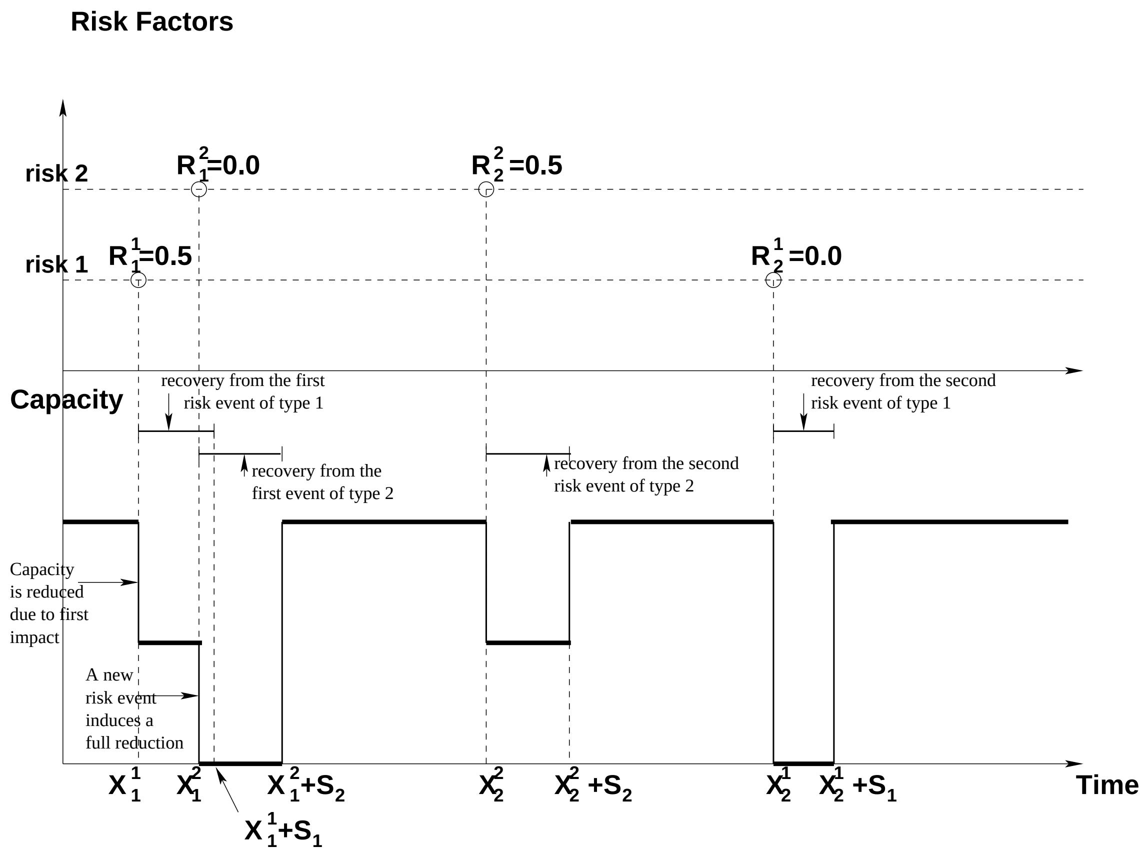

Fig. 11. A Sample Path for the Production Rate and Arrival of Risk Events.

capacity levels from both risk events. We can express $P(t)$ as

$$P(t) = C \cdot (\sum_{j=1}^{\infty} R_j^1 \cdot 1_{\{t-S_1<X_j^1\leq t\}}) \cdot (\sum_{j=1}^{\infty} R_j^2 \cdot 1_{\{t-S_2<X_j^2\leq t\}}). \tag{6.1}$$

A sample path for $P(t)$ in 6.1 is displayed in Figure 11.

The insurance company assumes the risk of inability to fulfill the demand due to reduction in capacity. Thus, the insurance company is not responsible from unsatisfied demand due to insufficient full capacity. Without loss of generality, the contract horizon spans only one time period. The production schedule is made before the contract horizon with a target production rate of D. If we set $C \geq D$, then there is no risk of deviating from the target production rate as long as the facility operates under full capacity. Then the lost sales during the contract horizon is max $(D - \int_0^1 P(t)dt, 0)$.

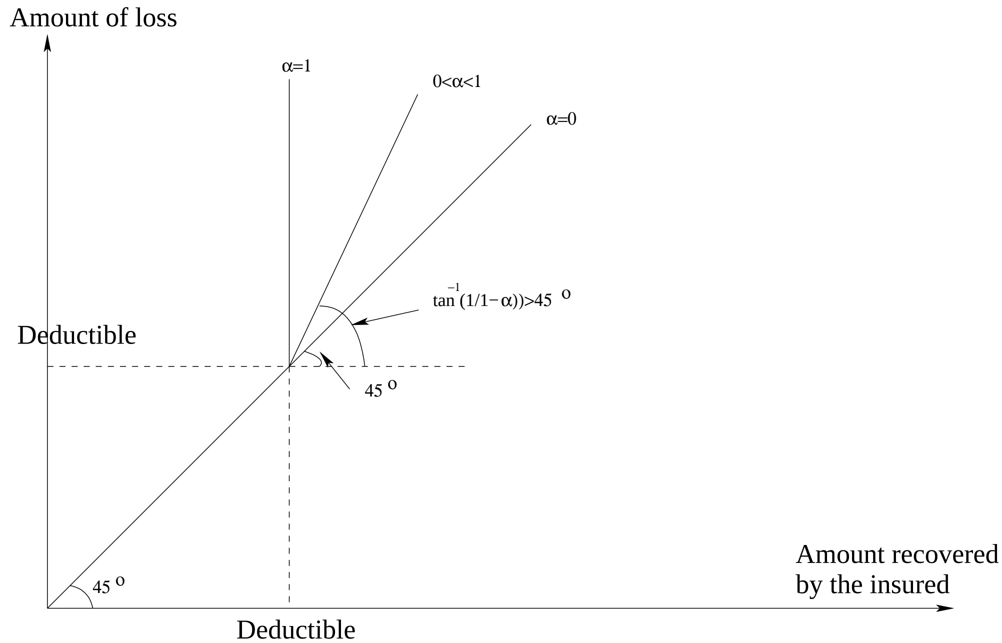

Fig. 12. An Ilustration of Coinsurance Contracts as a Function of α.

Note the implicit assumption that the total demand of D during the contract horizon can only be fulfilled from production during that period, i.e. inventory level at the beginning of the contract horizon is zero.

The insurance company offers a coinsurance contract $(\pi(\cdot), \alpha)$ with zero deductible and α is the fraction of the actual deviation from the target production level that the insurance company guarantees to pay. For example, if there is a damage of \$5 million and $\alpha = 0.8$, then the insurance company pays \$4 million. Figure 12 illustrates the relation between amount recovered and amount of loss as a function of α. The level of premium $\pi(\cdot)$ depends on the risk pattern over the contract horizon, α, the safety load factor θ and the cost per unit of lost production κ. The insurance premium takes the form

$$\pi(\alpha, \kappa, \theta) = (1 + \theta) \cdot \alpha \cdot \kappa \cdot \mathbb{E}\,[\max\,(D - \int_0^1 P(t)dt, 0)]. \tag{6.2}$$

Note that $\pi(\cdot)$ is linear in α and θ. The decision maker must choose the optimal level of the risk sharing percentage, α for the insurance contract given an information endowment. An information endowment involves all the events that the decision maker observes before making the final decision. In particular, an information endowment can include historical data on risk events experienced by the firm making the decision or other firms operating in a similar environment. Information endowment can also include severity assessments if a specific risk should occur. This information can either be gathered internally by risk experts, or purchased by external experts. The decision maker may observe these events before making the decision. In this chapter, we focus on the insurance contract decision, and also ask the question of obtaining the"right" information endowment to design a "better" insurance contract. We will have more to say about modeling the information endowment in what follows.

The level of wealth which includes all the assets of the firm is denoted by W. Let $u(x) : \mathbb{R} \to \mathbb{R}$ be the utility function of the firm. We will assume $u(\cdot)$ is concave, increasing and exhibiting non-increasing degree of risk aversion. This last assumption states that a decision maker with a higher wealth cares less about losing a fixed amount than a decision maker with a lower wealth. We only consider the risks that can be translated into financial terms, so the firm's utility is assumed to depend only on money, i.e., net profitability or some other suitable financial metric.

When the firm does not acquire any information, the insurance problem can be stated as

$$\max_{\alpha}\{P(E_d^c)\cdot\mathbb{E}\ [u(W-\pi+(p-c)\cdot D)|E_d^c]+ \\ P(E_d)\cdot\mathbb{E}\ [u(W-\pi+(p-c)\cdot(\alpha\cdot D+(1-\alpha)\cdot\int_0^1 P(t)dt))|E_d]\} \quad (6.3)$$

where $E_x = \{\int_0^1 P(t)dt \leq x\}$. Since, $\mathbb{E}\ [u(W-\pi+(p-c)\cdot D)|E_d^c] = u(W-\pi+(p-c)\cdot D)$, 6.3 reduces to

$$\max_{\alpha}\{P(E_d^c)\cdot u(W-\pi+(p-c)\cdot D)+ \\ P(E_d)\cdot\mathbb{E}[u(W-\pi+(p-c)\cdot(\alpha\cdot D+(1-\alpha)\cdot\int_0^1 P(t)dt))|E_d]\}$$

Let $\mathcal{F}^*$ be the set of complete sub-σ-fields of $\mathcal{F}$. Information is a collection of events which is also a subset $\mathcal{F}^*$, hereafter called an information bundle. The firm has an option to purchase information bundles before time $t = 0$, i.e. before making the decision. The history of the risk process in time interval $(t_1, t_2]$, where $t_1, t_2 \in (-\infty, 1]$, is summarized in the information bundle $\mathcal{F}_{(t_1,t_2]}$. Conceptually, information bundles of the form $\mathcal{F}_{(t_1,t_2]}$ provide complete information about the process in a given time period. The history of the risk process up to time t is recorded in $\mathcal{F}_{(-\infty,t]}$ which we denote in short by $\mathcal{F}_t$. Note that, $\{\mathcal{F}_t\}_{t\in(-\infty,1]}$ is an increasing collection of σ-algebras.

However, complete information about the period $[0, 1]$ is not available and it might be costly to gather complete information about the history of the process in some time period $(t_1, t_2]$ for $t_1, t_2 \leq 0$. In practice, most of the available information bundles provide partial information. We will let $\mathcal{F}^a_{(t_1,t_2]}$ be the set of available information bundles that contain information related with the behavior of the risk process in $(t_1, t_2]$. If $\mathcal{F}_{(t_1,t_2]} \in \mathcal{F}^a_{(t_1,t_2]}$, then we say that history of the process can be completely revealed. The set of all available information bundles is $\mathcal{F}^a = \cup_{t_1\leq t_2; t_1,t_2\in(-\infty,1]} \mathcal{F}^a_{(t_1,t_2]}$. A very simple element of $\mathcal{F}^a_{(t_1,t_2]}$ can be an information bundle generated by a single event. For example, if the decision maker would like to learn whether the severity of risk events of type j in time interval $(t_1, t_2]$ lies in $[r_1, r_2]$ or not, then he needs to purchase the information bundle generated by the event $A = \{\omega : R^j_i \in [r_1, r_2] \ \forall i \text{ s.t. } X^j_i \in (t_1, t_2]\}$. Clearly, $\sigma(A) \subset \mathcal{F}_{(t_1,t_2]}$, and $\sigma(A) \in \mathcal{F}^a_{(t_1,t_2]}$. An important property of $\mathcal{F}^a$ is that if $\mathcal{G}_1, \mathcal{G}_2 \in \mathcal{F}^a$, then $\mathcal{G}_1 \vee \mathcal{G}_2 \in \mathcal{F}^a$.

There are several different approaches to assess the value of risk information. It has been shown that all the approaches do not lead to the same preference ranking of information bundles. We will employ the expected utility approach, which ranks the information bundles based on the marginal increase in expected utility after acquiring a particular information bundle. In this regard, the value of information is defined as

$$V(\mathcal{G},u) = \mathbb{E}\left\{\max_{\alpha}(P(E_d^c|\mathcal{G})\cdot u(W-\pi+(p-c)\cdot D-q(\mathcal{G}))+P(E_d|\mathcal{G})\cdot\right.$$
$$\mathbb{E}\left[u(W-\pi+(p-c)\cdot(\alpha\cdot D+(1-\alpha)\cdot\int_0^1 P(t)dt)-q(\mathcal{G}))|E_d,\mathcal{G}\right]\}-$$
$$\max_{\alpha}\{P(E_d^c)\cdot u(W-\pi+(p-c)\cdot D)+P(E_d)\cdot\mathbb{E}\left[u(W-\pi+\right.$$
$$\left.(p-c)\cdot(\alpha\cdot D+(1-\alpha)\cdot\int_0^1 P(t)dt))|E_d\right]\} \tag{6.4}$$

where $q(\mathcal{G})$ is the cost of information bundle $\mathcal{G}\in\mathcal{F}^a$.

Following 6.4, the decision maker's problem is stated as

$$\sup_{\mathcal{G}\in\mathcal{F}^a} V(\mathcal{G},u)$$

Hence, the decision maker chooses the information bundle that will yield the maximal increase in the expected utility with respect to the baseline case of no information.

VI.2. Numerical Results

As we mentioned in Chapter V, it is very difficult to obtain analytical results in insurance decision problems. Even if we assume an exponentially distributed amount of time between the end of recovery from a risk event and the next risk event, the computations are quite complex. Therefore, we simulate a number of scenarios to see the behavior of value of information with respect to its determinants. We consider a decision maker with the utility function

$$u(x) = \begin{cases} \sqrt{x+1}-1 & : \quad x \geq 0 \\ \frac{1}{16}-(x-\frac{1}{4})^2 & : \quad x < 0 \end{cases}$$

as in Chapter IV. The time between the end of revovery from a risk event and the next risk event is assumed to be exponentially distributed, with different rates for

Table 11. Parameter Values for the Simulation.

Price	5	λ_1	2	Wealth	10000
Cost	3	λ_2	5	Recovery Period 1	0.1
Demand	9000	Safety Load	0	Recovery Period 2	0.05

each type j, denoted by λ_j. Without loss of generality, we assume that the contract period is one time unit. The decision maker is assumed to know the recovery status of the production process at time 0. Under all the scenarios we discuss, the production process is assumed to face no recovery at time 0 from a risk event that occured prior to time 0.

We consider information that pertains to different periods during the contract horizon. Specific information bundles that we consider are $\mathcal{F}^{r,j}_{(0,t]}$ each of which is generated by the event $A^{r,j}_{(0,t]} = \{\omega : R^j_i \geq r \; \forall i \text{ st } X^j_i \in (0,t]\}$. In other words, information about the event $A^{r,j}_{(0,t]}$ answers the question of whether a risk event of type j occurs in $(0,t]$ or not, and if a risk event occurs, whether the severity exceeds r or not. Note that, r denotes the fractional remaining capacity, so the decision maker learns if the fractional decrease in the production rate is within a certain range or not even when a risk event hits the facility. In this respect, $A^{1,j}_{(0,t]}$ does not give any information about the risk event should a risk event occur, so this type of an event simply answers the question of whether an event occurs of type j in $(0,t]$.

The parameter values that we used in the simulation is illustrated in Table 11. In the following sections, we discuss the effect of changes of some of these parameters on the value of information. The maximum production rate is taken as 10000, so the demand is certainly satisfied in case no risk event hits the facility. Under this parameter setting the decision maker decides to purchase full coverage, attaining expected utility

Table 12. Value of Information about $A^{1,1}_{(0,t]}$ as a Function of t.

t	**VOI**	**t**	**VOI**	**t**	**VOI**	**t**	**VOI**
0.05	0	0.3	0.456768	0.55	0.590742	0.8	0.487168
0.1	0.025389	0.35	0.509922	0.6	0.587552	0.85	0.465086
0.15	0.172653	0.4	0.546433	0.65	0.580671	0.9	0.44066
0.2	0.29145	0.45	0.572242	0.7	0.567853	0.95	0.4136
0.25	0.384204	0.5	0.585856	0.75	0.549918	1	0.371465

of 162.105. Note that, all the simulations are replicated so as to achieve convergence of each result at least in two decimal places.

VI.2.1. Information on the Time of Risk Events

We first discuss the the value of information about events $A^{1,1}_{(0,t]}, A^{1,2}_{(0,t]}$ for different values of t. Tables 12 and 13 illusrate the behavior with respect to risk events 1 and 2 respectively. Maximum value of information is attained when $t \in (0.5, 0.6)$ for the first type of risk event and when $t \in (0.2, 0.3)$. For smaller values of t, information is not valuable as the decision maker still purchases full coverage after learning that there is no risk event for such a short period of time. For relatively lower values of t, the value of information has an increasing trend simply because the decision maker is able to reduce uncertainty for a longer period of time. Note that, the decision maker changes the decision when he learns that no risk event occurs in that specified time period. Hence, information becomes valuable as the decision maker reduces the purchase of insurance. As t gets larger, the likelihood of this valuable piece decreases, so value of information decreases after attaining some maximum value. The reason is that we evaluate the value of information a priori, and the event that induces a decision change

Table 13. Value of Information about $A^{1,2}_{(0,t]}$ as a Function of t.

t	**VOI**	**t**	**VOI**	**t**	**VOI**	**t**	**VOI**
0.05	0	0.3	0.221439	0.55	0.12512	0.8	0.04905
0.1	0.062521	0.35	0.208626	0.6	0.10625	0.85	0.03997
0.15	0.157648	0.4	0.188865	0.65	0.089193	0.9	0.032747
0.2	0.205712	0.45	0.15771	0.7	0.07323	0.95	0.027864
0.25	0.223573	0.5	0.145796	0.75	0.062088	1	0.022106

becomes less likely. For example, for the first risk type, the value of information about the event $A^{1,1}_{(0,0.9]}$ is less than that of $A^{1,1}_{(0,0.8]}$ although the decision maker attains a higher expected utility *after* learning that $A^{1,1}_{(0,0.9]}$ occurs rather than $A^{1,1}_{(0,0.8]}$ occurs. The same explanation applies to the second risk event, too.

If we compare the value of information with respect to two different types of risk events, we observe that the value of information is greater for the second risk event type for small values of t. This is not the case for larger values of t. The decision maker is always better of *after* learning that $A^{1,2}_{(0,t]}$ occurs rather than $A^{1,1}_{(0,t]}$ occurs because the second event exposes a more frequent risk. However, the likelihood of no risk event in a specific time period is considerably lower as t increases, so the decision maker prefers learning about the less frequent risk hoping that there is a greater chance of changing the original decision. Since the second risk event occurs 2.5 times more frequent on the the average than the first risk event, the second effect becomes dominant for fairly low values of t.

VI.2.2. Incorporation of Information on the Severity of Risk Events

In this section, we suppose that the decision maker may get information about

Table 14. Value of Information about $A^{r,1}_{(0,t]}$ as a Function of r and t.

t	**r**	**VOI**	**t**	**r**	**VOI**	**t**	**r**	**VOI**
0.25	0.2	0	0.5	0.2	0.0546	0.75	0.2	0.220506
0.25	0.35	0.01537	0.5	0.35	0.28914	0.75	0.35	0.485925
0.25	0.5	0.139744	0.5	0.5	0.450696	0.75	0.5	0.626705
0.25	0.65	0.24435	0.5	0.65	0.550528	0.75	0.65	0.672948
0.25	0.8	0.327278	0.5	0.8	0.597733	0.75	0.8	0.652224
0.25	1	0.384204	0.5	1	0.585856	0.75	1	0.549918

the risk events. The value of information is increasing in r for small values of r, whereas for larger values of r, it is decreasing in r. Tables 14 and 15 illustrate the behavior of the value of information for risk events of type 1 and 2, respectively. The decision maker is always better off *after* learning that $A^{r,j}_{(0,t]}$ occurs for larger values of r. The decision maker's original decision is to purchase full coverage. The only way to change the decision is learning either no risk event occurs in some given time frame or learning that even if risk events occur, they are not going to remove a big fraction of the capacity anyway. For sufficiently small r, the decision maker may choose to abandon insurance. However, the likelihood of $A^{r,j}_{(0,t]}$ is decreasing with r and t, hence as we evaluate information a priori the positive effect of $A^{r,j}_{(0,t]}$ is lessened.

Suppose $A^{r,j}_{(0,t]}$ occurs. The decision maker still purchases full coverage for small values of r and t. This implies that the value of information is zero. It is very probable that $A^{r,j}_{(0,t]}$ occurs for small values of r and t. Therefore, the optimal action when $A^{r,j}_{(0,t]}$ occurs is likely to be chosen when no information is obtained. For large values r and t, this is not the case as, so the information becomes valuable.

Table 15. Value of Information about $A^{r,2}_{(0,t]}$ as a Function of r and t.

t	**r**	**VOI**	**t**	**r**	**VOI**	**t**	**r**	**VOI**
0.25	0.2	0	0.5	0.2	0.079618	0.75	0.2	0.19581
0.25	0.35	0.04241	0.5	0.35	0.22795	0.75	0.35	0.289332
0.25	0.5	0.14014	0.5	0.5	0.271425	0.75	0.5	0.26125
0.25	0.65	0.19913	0.5	0.65	0.25875	0.75	0.65	0.193563
0.25	0.8	0.226968	0.5	0.8	0.2166	0.75	0.8	0.126788
0.25	1	0.223573	0.5	1	0.145796	0.75	1	0.062088

VI.2.3. Effect of Profit, Wealth Level and Safety Load on the Value of Information

In this analysis, we ignore all the physical damage to the facility when a risk event is observed. The facility is insured against loss sales. Therefore, the profit of the product produced in the facility increases the loss should a risk event occur. As illustrated in Tables 16 and 17, the value of information increases as price increases. The decision maker is more willing to reduce uncertainty as the profitability of the product increases. Note that, the evaluations are made only on events $A^{1,j}_{(0,t]}$, i.e., the decision maker does not learn any information about the severity of the risk events.

This line of reasoning does not work for very profitable products, though. As the prices increase, the decision maker favors purchasing of insurance even when he learns that no event occurs in a particular time period. In this case, the value of information should decrease. Tables 16 and 17, does not reflect this, as this second effect starts dominating in a slow manner. Note that, the profitability of the product does not have an effect on the relation between t and the value of information.

Table 16. Value of Information about $A^{1,1}_{(0,t]}$ as a Function of Price and t.

t	**p**	**VOI**	**t**	**p**	**VOI**	**t**	**p**	**VOI**	**t**	**p**	**VOI**
0.2	5	0.29145	0.4	5	0.546433	0.6	5	0.587552	0.8	5	0.487168
0.2	8	0.50853	0.4	8	0.973432	0.6	8	1.050791	0.8	8	0.950208
0.2	10	0.61372	0.4	10	1.183115	0.6	10	1.277444	0.8	10	1.155844
0.2	18	0.92594	0.4	18	1.797347	0.6	18	1.943858	0.8	18	1.759824
0.2	30	1.25759	0.4	30	2.44705	0.6	30	2.649703	0.8	30	2.398952
0.2	50	1.66964	0.4	50	3.255699	0.6	50	3.526817	0.8	50	3.193216
0.2	80	2.14065	0.4	80	4.188272	0.6	80	4.534565	0.8	80	4.106862
0.2	100	2.40463	0.4	100	4.705071	0.6	100	5.098037	0.8	100	4.6157

In this document, we have mentioned several times that the value of information does not have any general monotonic relation with its determinants. In this respect, we do not claim that the monotonic relationships that are illustrated in tables can be generalized in any manner. Tables 18 and 19 shows that value of information about events $A^{1,j}_{(0,t]}$ is decreasing in wealth. As the decision maker displays decreasing risk aversion, the loss becomes less hazardous for the decision maker as wealth increases. However, we need a more thorough analysis to see how general this conclusion is.

The cost of insurance depends on the safety load. In the original parameter settings, safety load is zero, which implies that the insurer is risk neutral. As safety load increases, the risk neutrality is lost and the insurer charges higher than the fair premium. In this respect, safety load also controls how risk averse the insurer is. Clearly, as the cost of insurance increases, the decision maker reduces the level of coverage. However, this does not necessarily imply that the value of information is lower. This

Table 17. Value of Information about $A^{1,2}_{(0,t]}$ as a Function of Price and t.

t	**p**	**VOI**	**t**	**p**	**VOI**	**t**	**p**	**VOI**	**t**	**p**	**VOI**
0.2	5	0.205712	0.4	5	0.188865	0.6	5	0.10625	0.8	5	0.04905
0.2	8	0.361376	0.4	8	0.336555	0.6	8	0.1899	0.8	8	0.08775
0.2	10	0.437184	0.4	10	0.40878	0.6	10	0.2308	0.8	10	0.106704
0.2	18	0.660192	0.4	18	0.621405	0.6	18	0.3512	0.8	18	0.162414
0.2	30	0.896816	0.4	30	0.846315	0.6	30	0.47865	0.8	30	0.221418
0.2	50	1.190848	0.4	50	1.126035	0.6	50	0.63705	0.8	50	0.294678
0.2	80	1.529408	0.4	80	1.447605	0.6	80	0.819	0.8	80	0.378954
0.2	100	1.718928	0.4	100	1.62729	0.6	100	0.92065	0.8	100	0.42597

is illustrated in Table 20. As the decision maker purchases partial coverage when safety load factor is low, the decision is sure to be changed when $A^{1,1}_{(0,t]}$ or ${A^{1,1}_{(0,t]}}^c$ occurs. Therefore, information becomes very valuable. For higher values of the safety load, the decision maker reduces the coverage to zero, no matter which of these two events occur, so information is valueless. The second effect of costly insurance becomes effective when the safety load is high enough to induce the decision maker to purchase no coverage when no information is acquired. A similar picture ensues for information on $A^{1,2}_{(0,t]}$.

Our analysis is far from being complete. While it is not possible to guess in advance which information bundle is the most valuable for the decision maker, it can easily be argued that information becomes very valuable when the decision maker initially purchases partial coverage. However, in most cases, the decisions are clear cut in that the decision maker purchases either full or no coverage.

Table 18. Value of Information about $A^{1,1}_{(0,t]}$ as a Function of Wealth and t.

t	**w**	**VOI**	**t**	**w**	**VOI**	**t**	**w**	**VOI**
0.25	1000	0.456924	0.5	1000	0.711712	0.75	1000	0.671007
0.25	5000	0.420564	0.5	5000	0.646944	0.75	5000	0.608344
0.25	10000	0.380204	0.5	10000	0.585856	0.75	10000	0.549918
0.25	15000	0.356328	0.5	15000	0.53912	0.75	15000	0.505541
0.25	20000	0.333906	0.5	20000	0.502688	0.75	20000	0.470753
0.25	30000	0.298152	0.5	30000	0.446752	0.75	30000	0.418125
0.25	50000	0.252702	0.5	50000	0.37536	0.75	50000	0.350556
0.25	80000	0.211494	0.5	80000	0.3128	0.75	80000	0.291684
0.25	100000	0.192708	0.5	100000	0.284832	0.75	100000	0.265593

Table 19. Value of Information about $A^{1,2}_{(0,t]}$ as a Function of Wealth and t.

t	**w**	**VOI**	**t**	**w**	**VOI**	**t**	**w**	**VOI**
0.25	1000	0.26691	0.5	1000	0.17712	0.75	1000	0.07572
0.25	5000	0.245098	0.5	5000	0.160884	0.75	5000	0.068664
0.25	10000	0.223573	0.5	10000	0.145796	0.75	10000	0.062088
0.25	15000	0.206927	0.5	15000	0.134234	0.75	15000	0.05712
0.25	20000	0.193725	0.5	20000	0.125132	0.75	20000	0.053184
0.25	30000	0.172774	0.5	30000	0.111192	0.75	30000	0.047232
0.25	50000	0.146083	0.5	50000	0.093398	0.75	50000	0.039624
0.25	80000	0.122262	0.5	80000	0.077818	0.75	80000	0.032976
0.25	100000	0.111356	0.5	100000	0.070848	0.75	100000	0.030048

Table 20. Value of Information about $A^{1,1}_{(0,t]}$ as a Function of Safety Load and t.

t	**safety load**	**VOI**	**t**	**safety load**	**VOI**	**t**	**safety load**	**VOI**
0.25	0	0.384	0.5	0	0.586	0.75	0	0.55
0.25	0.1	0.601	0.5	0.1	0.7	0.75	0.1	0.602
0.25	0.2	0.434	0.5	0.2	0.43	0.75	0.2	0.27
0.25	0.3	0.263	0.5	0.3	0.157	0.75	0.3	0
0.25	0.4	0.092	0.5	0.4	0	0.75	0.4	0
0.25	0.5	0	0.5	0.5	0	0.75	0.5	0

CHAPTER VII

CONCLUSION

In this chapter, we present a summary of the research and highlight the contributions of our results. We also provide a discussion on possible future directions.

VII.1. Summary and Contributions

Every decision is made in the light of some information bundle that is possessed at the decision epoch. The primary objective of this research is to determine the value of information bundles available to an expected utility maximizer and to determine how to rank the information bundles for a given decision maker. There are two main motivations behind this objective. First, the impact of information on decisions has not been studied in detail in engineering decision problems. The models do not address the question of the choice of optimal information bundle to improve the decision. While some work has been done to evaluate the value of specific information bundles under specific scenarios for risk neutral decision makers, no work has been done to address the ranking of a general class of information bundles for a general class of decision makers. Second, information gathering plays a key role in reducing uncertainty in enterprise risk management decisions. As new markets emerge and corporations begin to move their facilities abroad, they are exposed to physical hazard risks about which they may have minimal information. In order to improve their risk transfer decisions, the corporations consider acquision of information about the new environment usually at significant cost. Due to budget limitations, the decision maker must determine the most valuable information for risk management purposes.

In this research, we first addressed the question of value of information in simple lotteries. Simple lotteries provide an easy and a versatile way to model decisions under uncertainty with a finite number of possible outcomes. As we have shown, the results are extended to general lotteries in a trivial manner in case the number of actions chosen optimal given different outcomes is finite. We computed value of information using the expected utility approach for a class of information bundles due to computational ease. We established the relation between the risk attitude and ranking of information as well as the relation between the value of information and the level of initial wealth for a class of utility functions and information bundles. All of our results were limited to certain classes of utility functions and information bundles because of the lack of a general monotonic relation between the determinants of the value of information and the value of information.

In Chapter IV, the buying price of information is studied. The decision maker is assumed to pay for information before making the decision, and the buying price is the maximum amount the decision maker is willing to pay for information. It is known however that the expected utility approach we employed in Chapter III does not always induce the same ranking of information with the buying price approach. We have characterized the relation between these two approaches in detail for a class of information bundles. Furthermore, we established the relation between the buying price and the risk attitude of the decision maker.

We study static and dynamic insurance decisions in Chapters V and VI. Chapter V considers a static insurance decision problem where the decision maker purchases coinsurance, and considers acquisition of information before making the decision. It is not possible to obtain closed form expressions for the value of information, even for the expected utility approach, so we used a particular form of a utility function for demonstrating ranking of different information bundles. For the dynamic insurance decision problem, we considered a decision maker considering insurance against business interruption risk in a facility exposed to multiple risk factors. We build a simulation model to compute the value of information on severity of risk events and the time of the risk events. The model provided preliminiary results for a real life problem at GM, in which the company seeks information to reduce uncertainty about physical risks that their facilities are exposed to. In this respect, there is a nice interplay between our results for both the static and dynamic insurance decisions and enterprise risk management.

VII.2. Directions for Future Research

The main objective of this research is to quantify information in engineering decision problems. To this end, we present some general and model specific results that evaluates information bundles and establishes their preference orderings for different decision makers. We restricted our results in general to specific classes of information bundles and a specific class of utility functions. It is our intention to obtain a general result for value of information bundles using expected utility approach for a class of utility functions, removing the restriction on the information bundles. Uniform continuity of the value of information map with respect to the metrics proposed by Zandt [38] and Boylan [6] has been established. We think it is possible to find a metric that ranks the information bundles by comparing their similarity to a target information bundle, i.e. the information bundle that provides perfect information for a class of utility functions. Ohlson [30] offers a similar result, he shows that identical ranking of information bundles for a class of utility functions exists. However, he does not address the question of what does the similarity of two information bundles imply with respect to their value to the decision maker. Such a result could enable us to rank information bundles independent of the specific utility function, as long as the utility function is a member of the specific class we consider.

Our results for lotteries are far from being complete. We restricted our analysis to a specific class of information bundles, especially in Chapter IV. We plan to extend our analysis to more general information bundles and analyze the buying price of the more complex information bundles. Futhermore, we believe it is possible to construct the relation for the value of information using expected utility approach between similar information bundles, i.e., computation of the value of information bundle can be computed using the value of a similar information bundle. This will help us compute the value of information for more complex information bundles.

As we mentioned earlier, our results have applications in enterprise risk management. The case study in Chapter VI offered a simple model for risk events that hit a single facility producing a single product serving a single product. We would like to extend that model to include multiple facilities serving multiple markets producing multiple products in that each product is priced differently in each market and service cost to each market is unequal. The numerical results were taken using a single utility function. However, it is not possible to determine a single utility function that represents the preferences of the enterprise, so we plan to obtain the robustness of the result using different utility functions. An obvious extension is to model the time between the recovery from the last risk event and the time to next risk event using semi Markov processes. We believe the analytical results are difficult to obtain, so in many cases simulations will be crucial.

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